

Extremal functions for singular Trudinger-Moser inequalities in the entire Euclidean space

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Abstract

In a previous work (Int. Math. Res. Notices 13 (2010) 2394-2426), Adimurthi-Yang proved a singular Trudinger-Moser inequality in the entire Euclidean space \mathbb{R}^N ($N \geq 2$). Precisely, if $0 \leq \beta < 1$ and $0 < \gamma \leq 1 - \beta$, then there holds for any $\tau > 0$,

$$\sup_{u \in W^{1,N}(\mathbb{R}^N), \int_{\mathbb{R}^N} (|\nabla u|^N + \tau |u|^N) dx \leq 1} \int_{\mathbb{R}^N} \frac{1}{|x|^{N\beta}} \left(e^{\alpha_N \gamma |u|^{\frac{N}{N-1}}} - \sum_{k=0}^{N-2} \frac{\alpha_N^k \gamma^k |u|^{\frac{kN}{N-1}}}{k!} \right) dx < \infty,$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the area of the unit sphere in \mathbb{R}^N . The above inequality is sharp in the sense that if $\gamma > 1 - \beta$, all integrals are still finite but the supremum is infinity. In this paper, we concern extremal functions for these singular inequalities. The regular case $\beta = 0$ has been considered by Li-Ruf (Indiana Univ. Math. J. 57 (2008) 451-480) and Ishiwata (Math. Ann. 351 (2011) 781-804). We shall investigate the singular case $0 < \beta < 1$ and prove that for all $\tau > 0$, $0 < \beta < 1$ and $0 < \gamma \leq 1 - \beta$, extremal functions for the above inequalities exist. The proof is based on blow-up analysis.

Key words: singular Trudinger-Moser inequality, extremal function, blow-up analysis
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1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded smooth domain, $W_0^{1,N}(\Omega)$ be the usual Sobolev space. Denote $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the area of the unit sphere in \mathbb{R}^N . The famous Trudinger-Moser inequality [32, 20, 19, 26, 17] reads

$$\sup_{u \in W_0^{1,N}(\Omega), \int_{\Omega} |\nabla u|^N dx \leq 1} \int_{\Omega} e^{\alpha |u|^{\frac{N}{N-1}}} dx < \infty, \quad \forall \alpha \leq \alpha_N. \quad (1)$$

This inequality is sharp in the sense that all integrals are still finite when $\alpha > \alpha_N$, but the supremum is infinity. It was extended by Cao [4], J. M. do Ó [9], Panda [18], Ruf [21], and Li-Ruf

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[15] to the entire Euclidean space \mathbb{R}^N ($N \geq 2$). Namely

$$\sup_{u \in W^{1,N}(\mathbb{R}^N), \int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N) dx \leq 1} \int_{\mathbb{R}^N} \left(e^{\alpha |u|^{\frac{N}{N-1}}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{\frac{Nk}{N-1}}}{k!} \right) dx < \infty, \quad \forall \alpha \leq \alpha_N. \quad (2)$$

Recently, several interesting developments of (2) has been obtained by J. M. do Ó and M. de Souza [7, 10].

Using a rearrangement argument and a change of variables, Adimurthi-Sandeep [1] generalized the Trudinger-Moser inequality (1) to a singular version as follows:

$$\sup_{u \in W_0^{1,N}(\Omega), \int_{\Omega} |\nabla u|^N dx \leq 1} \int_{\Omega} \frac{e^{\alpha_N \gamma |u|^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx < \infty, \quad 0 \leq \beta < 1, \quad 0 < \gamma \leq 1 - \beta. \quad (3)$$

This inequality is also sharp in the sense that all integrals are still finite when $\gamma > 1 - \beta$, but the supremum is infinity. Obviously, if $\beta = 0$, then (3) reduces to (1). Later, (3) was extended to the entire \mathbb{R}^N by Adimurthi-Yang [3]. Precisely there holds for constants $\tau > 0$, $0 \leq \beta < 1$ and $0 < \gamma \leq 1 - \beta$,

$$\sup_{\int_{\mathbb{R}^N} (|\nabla u|^N + \tau |u|^N) dx \leq 1} \int_{\mathbb{R}^N} \frac{1}{|x|^{N\beta}} \left(e^{\alpha_N \gamma |u|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{(\alpha_N \gamma)^k |u|^{kN/(N-1)}}{k!} \right) dx < \infty. \quad (4)$$

Clearly, (2) is a special case of (4). It should be remarked that in [3], the proof of (4) is essentially based on the Young inequality; while in [15], (2) is proved via the method of blow-up analysis. Such kind of singular Trudinger-Moser inequalities are very important in analysis of partial differential equations, see for examples [27, 28, 29].

An interesting problem on Trudinger-Moser inequalities is whether or not extremal functions exist. Existence of extremal functions for the Trudinger-Moser inequality (1) was obtained by Carleson-Chang [5] when Ω is the unit ball, by M. Struwe [23] when Ω is close to the ball in the sense of measure, by M. Flucher and K. Lin [11, 16] when Ω is a general bounded smooth domain, and by Y. Li [14] for compact Riemannian surfaces. For recent developments, we refer the reader to Yang [30]. On extremal functions for (2), it was proved by Ruf [21] and Ishiwata [12] that if $N = 2$, then there exists some $\epsilon_0 > 0$ such that for all $\epsilon_0 < \alpha \leq 2\pi$, the supremum

$$\sup_{u \in W^{1,2}(\mathbb{R}^2), \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx \leq 1} \int_{\mathbb{R}^2} (e^{\alpha u^2} - 1) dx$$

can be attained by some function $u \in W^{1,2}(\mathbb{R}^2)$ satisfying $\|u\|_{W^{1,2}(\mathbb{R}^2)} \leq 1$. While for sufficiently small $\alpha > 0$, the above supremum can not be attained. If $N \geq 3$, then for any $0 \leq \alpha < \alpha_N$, the supremum in (2) can be achieved. While Li-Ruf [15] proved that when $\alpha = \alpha_N$, extremal function exists for the above supremum.

Our aim is to find extremal functions for the singular Trudinger-Moser inequality (4) in the case $0 < \beta < 1$. Note that the case $\beta = 0$ has been studied by Ruf [21], Ishiwata [12] and Li-Ruf [15]. While these two situations are quite different in analysis. Throughout this paper, we write for all $\tau \in (0, \infty)$,

$$\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^N} |\nabla u|^N dx + \tau \int_{\mathbb{R}^N} |u|^N dx \right)^{1/N}. \quad (5)$$

Obviously $\|\cdot\|_{1,\tau}$ is equivalent to the standard Sobolev norm on $W^{1,N}(\mathbb{R}^N)$. Define a function $\zeta : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\zeta(N, s) = e^s - \sum_{k=0}^{N-2} \frac{s^k}{k!} = \sum_{k=N-1}^{\infty} \frac{s^k}{k!}. \quad (6)$$

Our main results are the existence of extremal functions for subcritical or critical singular Trudinger-Moser inequality, which can be stated as the following two theorems respectively.

Theorem 1. (Subcritical case) Let $N \geq 2$, $\tau > 0$, $\|\cdot\|_{1,\tau}$ and $\zeta : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined as in (5) and (6) respectively. Then for any $0 < \beta < 1$ and $0 < \epsilon < 1 - \beta$, the supremum

$$\Lambda_{N,\beta,\tau,\epsilon} = \sup_{u \in W^{1,N}(\mathbb{R}^N), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta-\epsilon)|u|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \quad (7)$$

can be attained by some nonnegative decreasing radially symmetric function $u_\epsilon \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C^0(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ with $\|u_\epsilon\|_{1,\tau} = 1$.

Theorem 2. (Critical case) Let $N \geq 2$, $\tau > 0$, $\|\cdot\|_{1,\tau}$ and $\zeta : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined as in (5) and (6) respectively. Then for any $0 < \beta < 1$, the supremum

$$\Lambda_{N,\beta,\tau} = \sup_{u \in W^{1,N}(\mathbb{R}^N), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)|u|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \quad (8)$$

can be attained by some nonnegative decreasing radially symmetric function $u^* \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C^0(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N)$ with $\|u^*\|_{1,\tau} = 1$.

Trudinger-Moser inequalities involved the norm $\|\cdot\|_{1,\tau}$ was first introduced by Adimurthi-Yang [3]. This type of inequalities are easy to use in analysis of partial differential equations with exponential growth. It should be remarked that both the above inequalities and existence of extremal functions are independent of τ . Let us give the outline of proving Theorems 1 and 2. The proof of Theorem 1 is based on a direct method of variation. By a rearrangement argument, we can take a maximizing sequence u_j satisfying $u_j \geq 0$ and decreasing radially symmetric. Clearly $u_j \rightharpoonup u_\epsilon$ weakly in $W^{1,N}(\mathbb{R}^N)$ for some u_ϵ . Since $0 < \epsilon < 1 - \beta$ and $0 < \beta < 1$, for any $\nu > 0$, there exists sufficiently large $R > 0$ such that

$$\int_{|x|>R} \frac{\zeta(N, \alpha_N(1-\beta-\epsilon)|u_j|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx < \nu.$$

Since $\alpha_N(1-\beta-\epsilon) < \alpha_N(1-\beta)$, we have by the singular Trudinger-Moser inequality (4) that

$$\lim_{j \rightarrow \infty} \int_{|x| \leq R} \frac{\zeta(N, \alpha_N(1-\beta-\epsilon)|u_j|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \int_{|x| \leq R} \frac{\zeta(N, \alpha_N(1-\beta-\epsilon)|u_\epsilon|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx.$$

Then the conclusion of Theorem 1 follows from the above two estimates.

Following Li-Ruf [15] and thereby following closely Carleson-Chang [5], Ding-Jost-Li-Wang [8] and Adimurthi-Struwe [2], we prove Theorem 2 via the method of blow-up analysis. Particularly we divide the proof into several steps:

Step 1. For any $0 < \epsilon < 1 - \beta$, the supremum $\Lambda_{N,\beta,\tau,\epsilon}$ can be attained by some function u_ϵ (This is the content of Theorem 1 exactly). The Euler-Lagrange equation of u_ϵ is semi-linear elliptic when $N = 2$, or quasi-linear elliptic when $N \geq 3$;

Step 2. Denote $c_\epsilon = u_\epsilon(0) = \max_{\mathbb{R}^N} u_\epsilon$. If c_ϵ is a bounded sequence, then applying elliptic estimates to the equation of u_ϵ , we conclude that u_ϵ converges to a desired extremal function in $C^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \cap C^0_{\text{loc}}(\mathbb{R}^N)$. If $c_\epsilon \rightarrow +\infty$, then by a delicate analysis on u_ϵ , we derive

$$\Lambda_{N,\beta,\tau} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta-\epsilon)u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \leq \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0}.$$

Here $A_0 = \lim_{x \rightarrow 0} (G(x) + (N/\alpha_N) \log |x|)$, G is a Green function satisfying

$$-\operatorname{div}(|\nabla G|^{N-2} \nabla G) + \tau G^{N-1} = \delta_0 \quad \text{in } \mathbb{R}^N,$$

where δ_0 is a Dirac measure centered at 0.

Step 3. We construct a sequence of functions $\phi_\epsilon \in W^{1,N}(\mathbb{R}^N)$ satisfying $\|\phi_\epsilon\|_{1,\tau} = 1$ and if ϵ is sufficiently small, then

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)\phi_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx > \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0}.$$

Comparing *Steps 2* and *3*, we conclude that c_ϵ must be bounded and thus the existence of extremal function follows from elliptic estimates. It should be remarked that in *Step 2*, we shall use an estimate of Carleson-Chang [5]:

Lemma 3. *Let B_1 be the unit ball in \mathbb{R}^N , $v_\epsilon \in W^{1,N}_0(B_1)$ satisfy $\int_{B_1} |\nabla v_\epsilon|^N dx \leq 1$, and $v_\epsilon \rightharpoonup 0$ weakly in $W^{1,N}_0(B_1)$. Then*

$$\limsup_{\epsilon \rightarrow 0} \int_{B_1} (e^{\alpha_N |v_\epsilon|^{N/(N-1)}} - 1) dx \leq \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k}}.$$

Before ending this introduction, we mention Csato-Roy [6], Iula-Mancini [13] and Yang-Zhu [31] who studied the same topic in bounded planar domain or compact Riemannian surface. Throughout this paper, we do *not* distinguish sequence and subsequence, the reader can easily see it from the context. We denote a ball centered at 0 with radius r by B_r , $o_\epsilon(1) \rightarrow 0$ as $\epsilon \rightarrow 0$, $o_r(1) \rightarrow 0$ as $r \rightarrow 0$, and $o_R(1) \rightarrow 0$ as $R \rightarrow \infty$.

The remaining part of this paper is devoted to the proof of Theorems 1 and 2 and organized as follows: Since the proof is transparent in \mathbb{R}^2 , we show it in Section 2. In Section 3, we prove Theorems 1 and 2 in $N(\geq 3)$ dimensions.

2. Two dimensional case

When $N = 2$, extremal functions for subcritical singular Trudinger-Moser inequalities are distributional solutions of elliptic partial differential equations of second order. Compared with $N \geq 3$, analysis in two dimensions becomes much easier and transparent, so we deal with this case first.

2.1. Proof of Theorem 1

We rephrase Theorem 1 as below:

Theorem 4. *Let $\tau > 0$ and $0 < \beta < 1$ be fixed. Then for any $0 < \epsilon < 1 - \beta$, there exists some nonnegative decreasing radially symmetric function $u_\epsilon \in C^1(\mathbb{R}^2 \setminus \{0\}) \cap C^0(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2)$ satisfying $\|u_\epsilon\|_{1,\tau} = 1$ and*

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx = \Lambda_{2,\beta,\tau,\epsilon} = \sup_{u \in W^{1,2}(\mathbb{R}^2), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u^2} - 1}{|x|^{2\beta}} dx. \quad (9)$$

Proof. Let $\tau > 0$, $0 < \beta < 1$ and $0 < \epsilon < 1 - \beta$ be fixed. Suppose that \tilde{u} is the decreasing rearrangement of $|u|$. It is known that $\int_{\mathbb{R}^2} \tilde{u}^2 dx = \int_{\mathbb{R}^2} u^2 dx$, $\int_{\mathbb{R}^2} |\nabla \tilde{u}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx$ and

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)\tilde{u}^2} - 1}{|x|^{2\beta}} dx \geq \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u^2} - 1}{|x|^{2\beta}} dx.$$

Here we used the Hardy-Littlewood inequality in the last estimate. Therefore we have

$$\Lambda_{2,\beta,\tau,\epsilon} = \sup_{u \in \mathcal{S}} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u^2} - 1}{|x|^{2\beta}} dx,$$

where \mathcal{S} is a set consisting of all nonnegative decreasing radially symmetric functions $u \in W^{1,2}(\mathbb{R}^2)$ with $\|u\|_{1,\tau} \leq 1$. Take $u_j \in \mathcal{S}$ such that $\int_{\mathbb{R}^2} (e^{4\pi(1-\beta-\epsilon)u_j^2} - 1)/|x|^{2\beta} dx \rightarrow \Lambda_{2,\beta,\tau,\epsilon}$ as $j \rightarrow \infty$. Without loss of generality, we can assume that there exists some function $u_\epsilon \in W^{1,2}(\mathbb{R}^2)$ such that up to a subsequence, as $j \rightarrow \infty$, there holds $u_j \rightarrow u_\epsilon$ weakly in $W^{1,2}(\mathbb{R}^2)$, $u_j \rightarrow u_\epsilon$ in $L_{\text{loc}}^p(\mathbb{R}^2)$ for any $p > 0$ and $u_j \rightarrow u_\epsilon$ a.e. in \mathbb{R}^2 . Hence up to a set of zero measure, u_ϵ is nonnegative decreasing radially symmetric on \mathbb{R}^2 . Moreover, we have that $\|u_\epsilon\|_{1,\tau} \leq \limsup_{j \rightarrow \infty} \|u_j\|_{1,\tau} \leq 1$. Note that $0 < \beta < 1$ and $0 < \epsilon < 1 - \beta$. Given any $\nu > 0$, in view of the Trudinger-Moser inequality (2), there exists a sufficiently large $r > 0$ such that for all $u \in W^{1,2}(\mathbb{R}^2)$ with $\|u\|_{1,\tau} \leq 1$,

$$\frac{1}{r^{2\beta}} \int_{|x|>r} (e^{4\pi(1-\beta-\epsilon)u^2} - 1) dx \leq \frac{1}{r^{2\beta}} \int_{\mathbb{R}^2} (e^{4\pi(1-\beta-\epsilon)u^2} - 1) dx < \nu. \quad (10)$$

Since $u_j \rightarrow u_\epsilon$ in $L_{\text{loc}}^p(\mathbb{R}^2)$ for any $p > 0$, we have by using the mean value theorem,

$$\int_{|x| \leq r} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx = \lim_{j \rightarrow \infty} \int_{|x| \leq r} \frac{e^{4\pi(1-\beta-\epsilon)u_j^2} - 1}{|x|^{2\beta}} dx. \quad (11)$$

Combining (10) and (11), we obtain

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx - \nu \leq \limsup_{j \rightarrow \infty} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_j^2} - 1}{|x|^{2\beta}} dx \leq \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx + \nu.$$

Since ν is arbitrary, there holds

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_j^2} - 1}{|x|^{2\beta}} dx = \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx.$$

This leads to (9). Noting that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \int_{\mathbb{R}^2} \frac{e^{\frac{4\pi(1-\beta-\epsilon)u_\epsilon^2}{\|u_\epsilon\|_{1,\tau}^2}} - 1}{|x|^{2\beta}} dx,$$

we get the extremal function u_ϵ , which is nonnegative and decreasing radially symmetric, and satisfies $\|u_\epsilon\|_{1,\tau} = 1$. A straightforward calculation shows that u_ϵ satisfies the following Euler-Lagrange equation

$$\begin{cases} -\Delta u_\epsilon + \tau u_\epsilon = \frac{1}{\lambda_\epsilon} \frac{u_\epsilon e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} & \text{in } \mathbb{R}^2, \\ u_\epsilon > 0 & \text{in } \mathbb{R}^2, \\ \|u_\epsilon\|_{1,\tau} = 1, \\ \lambda_\epsilon = \int_{\mathbb{R}^2} |x|^{-2\beta} u_\epsilon^2 e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} dx. \end{cases} \quad (12)$$

Applying elliptic estimates to (12), we have $u_\epsilon \in C^1(\mathbb{R}^2 \setminus \{0\}) \cap C^0(\mathbb{R}^2)$. Here $u_\epsilon > 0$ follows from the classical maximum principle and the fact that $u_\epsilon(0) = \max_{\mathbb{R}^2} u_\epsilon$. This completes the proof of the theorem. \square

From now on, we prove Theorem 2 by using the method of blow-up analysis.

2.2. Elementary properties of u_ϵ

In view of the equation (12), it is important to know whether λ_ϵ has a positive lower bound or not. For this purpose, we have the following:

Lemma 5. *Let λ_ϵ be as in (12). Then there holds $\liminf_{\epsilon \rightarrow 0} \lambda_\epsilon > 0$.*

Proof. For any $u \in W^{1,2}(\mathbb{R}^2)$ with $\|u\|_{1,\tau} \leq 1$, we calculate by employing Theorem 4,

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)u^2} - 1}{|x|^{2\beta}} dx = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u^2} - 1}{|x|^{2\beta}} dx \leq \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx.$$

This leads to

$$\Lambda_{2,\beta,\tau} = \sup_{u \in W^{1,2}(\mathbb{R}^2), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)u^2} - 1}{|x|^{2\beta}} dx \leq \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx. \quad (13)$$

But one can easily see that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \Lambda_{2,\beta,\tau}. \quad (14)$$

Moreover, using the inequality $e^t \leq 1 + te^t$ for $t \geq 0$, we get

$$\lambda_\epsilon \geq \frac{1}{4\pi(1-\beta-\epsilon)} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx.$$

This together with (13) and (14) leads to

$$\liminf_{\epsilon \rightarrow 0} \lambda_\epsilon \geq \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi(1-\beta-\epsilon)} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx = \frac{\Lambda_{2,\beta,\tau}}{4\pi(1-\beta)} > 0.$$

This ends the proof of the lemma. \square

Denote $c_\epsilon = u_\epsilon(0) = \max_{\mathbb{R}^2} u_\epsilon$. If c_ϵ is bounded, then applying elliptic estimates to (12), we can find some $u^* \in W^{1,2}(\mathbb{R}^2)$ such that $u_\epsilon \rightarrow u^*$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^2)$. Clearly u^* is the desired extremal function satisfying

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)u^{*2}} - 1}{|x|^{2\beta}} dx = \sup_{u \in W^{1,2}(\mathbb{R}^2), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)u^2} - 1}{|x|^{2\beta}} dx.$$

Hence the proof of Theorem 2 terminates. In the following, we assume $c_\epsilon \rightarrow +\infty$. Since u_ϵ is bounded in $W^{1,2}(\mathbb{R}^2)$, we can assume without loss of generality, u_ϵ converges to u_0 weakly in $W^{1,2}(\mathbb{R}^2)$, strongly in $L_{\text{loc}}^q(\mathbb{R}^2)$ for any $q > 0$, and a.e. in \mathbb{R}^2 . Then we have the following:

Lemma 6. $u_0 \equiv 0$ and $|\nabla u_\epsilon|^2 dx \rightarrow \delta_0$ weakly in the sense of measure, where δ_0 denotes the Dirac measure centered at $0 \in \mathbb{R}^2$. Moreover, $u_\epsilon \rightarrow 0$ strongly in $L^p(\mathbb{R}^2)$ for all $p \geq 2$.

Proof. For any $a \geq 0$, $b \geq 0$ and $p \geq 1$, there holds ([27], Lemma 2.1)

$$(e^a - 1)^p \leq e^{pa} - 1. \quad (15)$$

Using $e^{a+b} - 1 = (e^a - 1)(e^b - 1) + (e^a - 1) + (e^b - 1)$, the Hölder inequality and (15), we estimate

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p u_\epsilon^2} - 1}{|x|^{2\beta p}} dx &\leq \int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p((1+\nu)(u_\epsilon - u_0)^2 + (1+\nu^{-1})u_0^2)} - 1}{|x|^{2\beta p}} dx \\ &= \int_{\mathbb{R}^2} \frac{(e^{\beta_\epsilon p(1+\nu)(u_\epsilon - u_0)^2} - 1)(e^{\beta_\epsilon p(1+\nu^{-1})u_0^2} - 1)}{|x|^{2\beta p}} dx \\ &\quad + \int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p(1+\nu)(u_\epsilon - u_0)^2} - 1}{|x|^{2\beta p}} dx + \int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p(1+\nu^{-1})u_0^2} - 1}{|x|^{2\beta p}} dx \\ &\leq \left(\int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p p_1(1+\nu)(u_\epsilon - u_0)^2} - 1}{|x|^{2\beta p}} dx \right)^{1/p_1} \left(\int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p p_2(1+\nu^{-1})u_0^2} - 1}{|x|^{2\beta p}} dx \right)^{1/p_2} \\ &\quad + \int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p(1+\nu)(u_\epsilon - u_0)^2} - 1}{|x|^{2\beta p}} dx + \int_{\mathbb{R}^2} \frac{e^{\beta_\epsilon p(1+\nu^{-1})u_0^2} - 1}{|x|^{2\beta p}} dx, \end{aligned} \quad (16)$$

where $\beta_\epsilon = 4\pi(1 - \beta - \epsilon)$, $p > 1$, $\nu > 0$, $p_1 > 1$ and $1/p_1 + 1/p_2 = 1$. We first prove that $u_0 \equiv 0$. Suppose not. Since

$$\|u_\epsilon - u_0\|_{1,\tau}^2 = \|u_\epsilon\|_{1,\tau}^2 + \|u_0\|_{1,\tau}^2 - 2 \int_{\mathbb{R}^2} (\nabla u_\epsilon \nabla u_0 + \tau u_\epsilon u_0) dx = 1 - \|u_0\|_{1,\tau}^2 + o_\epsilon(1),$$

one can choose p, p_1 sufficiently close to 1 and ν sufficiently close to 0 such that

$$\frac{\beta_\epsilon p p_1(1+\nu)\|u_\epsilon - u_0\|_{1,\tau}^2}{4\pi} + \frac{2\beta p}{2} < 1.$$

In view of the singular Trudinger-Moser inequality (4), we conclude that all integrals on the right hand side of (16) are bounded. Therefore

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)p u_\epsilon^2} - 1}{|x|^{2\beta p}} dx \leq C$$

for some constant C depending only on β and p . It follows that $e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}/|x|^{2\beta}$ is bounded in $L^p(B_1)$. This together with Lemma 5 and u_ϵ is bounded in $L^q(B_1)$ for all $q > 0$ implies that Δu_ϵ is bounded in $L^{p'}(B_1)$ for some $p' > 1$. Applying elliptic estimate to (12), we conclude that u_ϵ is uniformly bounded in $B_{1/2}$. This contradicts $c_\epsilon \rightarrow +\infty$. Therefore $u_0 \equiv 0$.

We next prove that $|\nabla u_\epsilon|^2 dx \rightarrow \delta_0$. For otherwise, we can choose sufficiently small $\bar{r} > 0$ such that

$$\int_{B_{\bar{r}}} (|\nabla u_\epsilon|^2 + \tau u_\epsilon^2) dx \leq \eta < 1$$

for sufficiently small $\epsilon > 0$. Hence Δu_ϵ is bounded in $L^{p''}(B_{\bar{r}})$ for some $p'' > 1$, and thus elliptic estimate leads to u_ϵ is uniformly bounded in $B_{\bar{r}/2}$ contradicting $c_\epsilon \rightarrow +\infty$.

For the last assertion, noting that $\|u_\epsilon\|_{1,\tau} = 1$ and $|\nabla u_\epsilon|^2 dx \rightarrow \delta_0$, we obtain $\|u_\epsilon\|_{L^2(\mathbb{R}^2)} = o_\epsilon(1)$. Taking $M > 0$ such that if $|x| > M$, then $u_\epsilon < 1$, one has for any $p > 2$,

$$\begin{aligned} \int_{\mathbb{R}^2} u_\epsilon^p dx &= \int_{|x|>M} u_\epsilon^p dx + \int_{|x|\leq M} u_\epsilon^p dx \\ &\leq \int_{|x|>M} u_\epsilon^2 dx + o_\epsilon(1) \\ &\leq \int_{\mathbb{R}^2} u_\epsilon^2 dx + o_\epsilon(1) = o_\epsilon(1). \end{aligned}$$

Here we have used the fact that $u_\epsilon \rightarrow 0$ in $L_{\text{loc}}^q(\mathbb{R}^2)$ for any $q > 0$. This completes the proof of the lemma. \square

2.3. Blow-up analysis

Set $r_\epsilon = \sqrt{\lambda_\epsilon} c_\epsilon^{-1} e^{-2\pi(1-\beta-\epsilon)c_\epsilon^2}$, $\psi_\epsilon(x) = c_\epsilon^{-1} u_\epsilon(r_\epsilon^{1/(1-\beta)} x)$ and $\varphi_\epsilon(x) = c_\epsilon(u_\epsilon(r_\epsilon^{1/(1-\beta)} x) - c_\epsilon)$. Then we have the following:

Lemma 7. (i) For any $\gamma < 2\pi(1-\beta)$, there holds $r_\epsilon e^{\gamma c_\epsilon^2} \rightarrow 0$ as $\epsilon \rightarrow 0$; (ii) $\psi_\epsilon \rightarrow 1$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^2)$; (iii) $\varphi_\epsilon \rightarrow \varphi$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^2)$, where $\varphi(x) = -\frac{1}{4\pi(1-\beta)} \log(1 + \frac{\pi}{1-\beta} |x|^{2(1-\beta)})$ and $\int_{\mathbb{R}^2} |x|^{-2\beta} e^{8\pi(1-\beta)\varphi} dx = 1$.

Proof. (i) By definition of r_ϵ , we have

$$\begin{aligned} r_\epsilon^2 e^{2\gamma c_\epsilon^2} &= c_\epsilon^{-2} e^{-4\pi(1-\beta-\epsilon-\frac{\gamma}{2\pi})c_\epsilon^2} \int_{\mathbb{R}^2} \frac{u_\epsilon^2 e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx \\ &\leq c_\epsilon^{-2} \int_{\mathbb{R}^2} \frac{u_\epsilon^2 e^{2\gamma u_\epsilon^2}}{|x|^{2\beta}} dx \\ &\leq c_\epsilon^{-2} \int_{\mathbb{R}^2} \frac{u_\epsilon^2 (e^{2\gamma u_\epsilon^2} - 1)}{|x|^{2\beta}} dx + c_\epsilon^{-2} \int_{\mathbb{R}^2} \frac{u_\epsilon^2}{|x|^{2\beta}} dx. \end{aligned} \quad (17)$$

By Lemma 6, we know that $\|u_\epsilon\|_{L^p(\mathbb{R}^2)} = o_\epsilon(1)$ for any $p \geq 2$. As an easy consequence, there holds for any $p \geq 2$

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{u_\epsilon^p}{|x|^{2\beta}} dx = 0. \quad (18)$$

Noting that $\gamma < 2\pi(1 - \beta)$, we can choose $p_1 > 1$ such that $\gamma p_1 < 2\pi(1 - \beta)$. In view of (4), (15) and (18), we have by the Hölder inequality,

$$\int_{\mathbb{R}^2} \frac{u_\epsilon^2(e^{2\gamma u_\epsilon^2} - 1)}{|x|^{2\beta}} dx \leq \left(\int_{\mathbb{R}^2} \frac{e^{2\gamma p_1 u_\epsilon^2} - 1}{|x|^{2\beta}} dx \right)^{1/p_1} \left(\int_{\mathbb{R}^2} \frac{u_\epsilon^{2p_2}}{|x|^{2\beta}} dx \right)^{1/p_2} = o_\epsilon(1), \quad (19)$$

where $1/p_2 + 1/p_1 = 1$. Inserting (18) and (19) into (17), we obtain $r_\epsilon e^{\gamma c_\epsilon^2} \rightarrow 0$ as $\epsilon \rightarrow 0$.

(ii) It can be easily checked that ψ_ϵ satisfies the equation

$$-\Delta \psi_\epsilon(x) = -\tau r_\epsilon^{2/(1-\beta)} \psi_\epsilon(x) + c_\epsilon^{-2} |x|^{-2\beta} \psi_\epsilon(x) e^{4\pi(1-\beta-\epsilon)(u_\epsilon^2(r_\epsilon^{1/(1-\beta)} x) - c_\epsilon^2)}. \quad (20)$$

Since $|\psi_\epsilon| \leq 1$, $u_\epsilon^2 \leq c_\epsilon^2$ and $r_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, we have by applying elliptic estimates to (20), $\psi_\epsilon \rightarrow \psi$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^2)$, where ψ is a bounded harmonic function on \mathbb{R}^2 . Then the Liouville theorem leads to $\psi \equiv \psi(0) = 1$.

(iii) A straightforward calculation shows

$$-\Delta \varphi_\epsilon(x) = -\tau c_\epsilon^2 r_\epsilon^{2/(1-\beta)} \psi_\epsilon(x) + |x|^{-2\beta} \psi_\epsilon(x) e^{4\pi(1-\beta-\epsilon)(1+\psi_\epsilon(x))\varphi_\epsilon(x)}. \quad (21)$$

Note that $\varphi_\epsilon(x) \leq 0 = \max_{\mathbb{R}^2} \varphi_\epsilon$. Applying elliptic estimates to (21), we conclude that $\varphi_\epsilon \rightarrow \varphi$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^2)$, where φ is a distributional solution to

$$\begin{cases} -\Delta \varphi(x) = |x|^{-2\beta} e^{8\pi(1-\beta)\varphi(x)} & \text{in } \mathbb{R}^2, \\ \varphi(0) = 0. \end{cases} \quad (22)$$

Since u_ϵ is decreasing symmetric and $u_\epsilon(0) = \max_{\mathbb{R}^2} u_\epsilon = c_\epsilon$, φ must be decreasing symmetric and $\varphi(0) = \max_{\mathbb{R}^2} \varphi$. If we set $\bar{\varphi}(r) = \varphi(x)$ for any $x \in \mathbb{R}^2$ and $r = |x|$, then (22) reduces to

$$\begin{cases} -(r\bar{\varphi}')' = r^{1-2\beta} e^{8\pi(1-\beta)\bar{\varphi}}, \\ \bar{\varphi}(0) = 0. \end{cases} \quad (23)$$

Clearly, this equation has a special solution

$$\bar{\varphi}(r) = -\frac{1}{4\pi(1-\beta)} \log\left(1 + \frac{\pi}{1-\beta} r^{2(1-\beta)}\right).$$

By the standard uniqueness result of the ordinary differential equation (23), we have

$$\varphi(x) = -\frac{1}{4\pi(1-\beta)} \log\left(1 + \frac{\pi}{1-\beta} |x|^{2(1-\beta)}\right), \quad x \in \mathbb{R}^2.$$

It follows that

$$\int_{\mathbb{R}^2} |x|^{-2\beta} e^{8\pi(1-\beta)\varphi} dx = \int_0^{+\infty} \frac{2\pi r^{1-2\beta}}{(1 + \frac{\pi}{1-\beta} r^{2(1-\beta)})^2} dr = \int_0^{+\infty} \frac{1}{(1+t)^2} dt = 1. \quad (24)$$

This completes the proof of the lemma. \square

Lemma 7 gives convergence behavior of u_ϵ near 0. To reveal the convergence behavior of u_ϵ away from 0, following [15], we define $u_{\epsilon,\gamma} = \min\{u_\epsilon, \gamma c_\epsilon\}$ for any $0 < \gamma < 1$. Then we have the following:

Lemma 8. For any $0 < \gamma < 1$, there holds $\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} |\nabla u_{\epsilon, \gamma}|^2 dx = \gamma$.

Proof. Testing the equation (12) by $u_{\epsilon, \gamma}$, we have for any fixed $R > 0$,

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u_{\epsilon, \gamma}|^2 dx &= -\tau \int_{\mathbb{R}^2} u_{\epsilon} u_{\epsilon, \gamma} dx + \frac{1}{\lambda_{\epsilon}} \int_{\mathbb{R}^2} u_{\epsilon} u_{\epsilon, \gamma} \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon}^2}}{|x|^{2\beta}} dx \\ &\geq \frac{1}{\lambda_{\epsilon}} \int_{B_{R_{\epsilon}^{1/(1-\beta)}}} \gamma c_{\epsilon} u_{\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon}^2}}{|x|^{2\beta}} dx + o_{\epsilon}(1) \\ &= (1 + o_{\epsilon}(1)) \gamma \int_{B_R} \frac{e^{8\pi(1-\beta)\varphi(x)}}{|x|^{2\beta}} dx + o_{\epsilon}(1). \end{aligned}$$

Hence

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} |\nabla u_{\epsilon, \gamma}|^2 dx \geq \gamma \int_{B_R} \frac{e^{8\pi(1-\beta)\varphi(x)}}{|x|^{2\beta}} dx.$$

In view of (24), passing to the limit $R \rightarrow +\infty$, we obtain

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} |\nabla u_{\epsilon, \gamma}|^2 dx \geq \gamma. \quad (25)$$

Testing the equation (12) by $(u_{\epsilon} - \gamma c_{\epsilon})^+$, we obtain for any fixed $R > 0$,

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla(u_{\epsilon} - \gamma c_{\epsilon})^+|^2 dx &= -\tau \int_{\mathbb{R}^2} u_{\epsilon} (u_{\epsilon} - \gamma c_{\epsilon})^+ dx + \int_{\mathbb{R}^2} (u_{\epsilon} - \gamma c_{\epsilon})^+ u_{\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon}^2}}{\lambda_{\epsilon}|x|^{2\beta}} dx \\ &\geq \frac{1}{\lambda_{\epsilon}} \int_{B_{R_{\epsilon}^{1/(1-\beta)}}} u_{\epsilon} (u_{\epsilon} - \gamma c_{\epsilon})^+ \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon}^2}}{|x|^{2\beta}} dx + o_{\epsilon}(1) \\ &= (1 + o_{\epsilon}(1))(1 - \gamma) \int_{B_R} \frac{e^{8\pi(1-\beta)\varphi(x)}}{|x|^{2\beta}} dx + o_{\epsilon}(1). \end{aligned}$$

Similarly as above, we have

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} |\nabla(u_{\epsilon} - \gamma c_{\epsilon})^+|^2 dx \geq 1 - \gamma. \quad (26)$$

Note that

$$\int_{\mathbb{R}^2} |\nabla u_{\epsilon, \gamma}|^2 dx + \int_{\mathbb{R}^2} |\nabla(u_{\epsilon} - \gamma c_{\epsilon})^+|^2 dx = \int_{\mathbb{R}^2} |\nabla u_{\epsilon}|^2 dx = 1 + o_{\epsilon}(1). \quad (27)$$

Combining (25), (26) and (27), we conclude the lemma. \square

Lemma 9. There holds

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon}^2} - 1}{|x|^{2\beta}} dx = \lim_{\epsilon \rightarrow 0} \frac{\lambda_{\epsilon}}{c_{\epsilon}^2}. \quad (28)$$

Proof. Let $0 < \gamma < 1$ be fixed. Using the inequality $e^t - 1 \leq te^t$ ($t \geq 0$) and the definition of

$u_{\epsilon,\gamma}$, we obtain

$$\begin{aligned}
\int_{u_\epsilon \leq \gamma c_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx &\leq \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon,\gamma}^2} - 1}{|x|^{2\beta}} dx \\
&\leq 4\pi(1-\beta) \int_{\mathbb{R}^2} \frac{u_{\epsilon,\gamma}^2 e^{4\pi(1-\beta-\epsilon)u_{\epsilon,\gamma}^2}}{|x|^{2\beta}} dx \\
&= 4\pi(1-\beta) \left\{ \int_{\mathbb{R}^2} u_{\epsilon,\gamma}^2 \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon,\gamma}^2} - 1}{|x|^{2\beta}} dx + \int_{\mathbb{R}^2} \frac{u_{\epsilon,\gamma}^2}{|x|^{2\beta}} dx \right\}. \quad (29)
\end{aligned}$$

It follows from (18) that

$$\int_{\mathbb{R}^2} \frac{u_{\epsilon,\gamma}^2}{|x|^{2\beta}} dx \leq \int_{\mathbb{R}^2} \frac{u_\epsilon^2}{|x|^{2\beta}} dx = o_\epsilon(1). \quad (30)$$

Moreover, combining Lemma 6 and Lemma 8, we have $\lim_{\epsilon \rightarrow 0} \|u_{\epsilon,\gamma}\|_{1,\tau}^2 = \gamma < 1$. Let $1 < p < 1/\gamma$ be fixed and $1/p + 1/p' = 1$. Using the Hölder inequality and the singular Trudinger-Moser inequality (4), we have

$$\begin{aligned}
\int_{\mathbb{R}^2} u_{\epsilon,\gamma}^2 \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon,\gamma}^2} - 1}{|x|^{2\beta}} dx &\leq \left(\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)pu_{\epsilon,\gamma}^2} - 1}{|x|^{2\beta}} dx \right)^{1/p} \left(\int_{\mathbb{R}^2} \frac{u_{\epsilon,\gamma}^{2p'}}{|x|^{2\beta}} dx \right)^{1/p'} \\
&\leq C \left(\int_{\mathbb{R}^2} \frac{u_{\epsilon,\gamma}^{2p'}}{|x|^{2\beta}} dx \right)^{1/p'} \quad (31)
\end{aligned}$$

for some constant C depending only on β , p and γ . Inserting (30) and (31) into (29), one has

$$\int_{u_\epsilon \leq \gamma c_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx = o_\epsilon(1). \quad (32)$$

Moreover, we estimate

$$\begin{aligned}
\int_{u_\epsilon > \gamma c_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx &= \int_{u_\epsilon > \gamma c_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx + o_\epsilon(1) \\
&\leq \frac{1}{\gamma^2} \int_{u_\epsilon > \gamma c_\epsilon} \frac{u_\epsilon^2}{c_\epsilon^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx + o_\epsilon(1) \\
&\leq \frac{1}{\gamma^2} \frac{\lambda_\epsilon}{c_\epsilon^2} + o_\epsilon(1). \quad (33)
\end{aligned}$$

Combining (32) and (33), we have

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \frac{1}{\gamma^2} \liminf_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^2}.$$

Letting $\gamma \rightarrow 1$, we conclude

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \liminf_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \quad (34)$$

On the other hand,

$$\begin{aligned}
\frac{\lambda_\epsilon}{c_\epsilon^2} &= \int_{\mathbb{R}^2} \frac{u_\epsilon^2}{c_\epsilon^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx \\
&= \int_{\mathbb{R}^2} \frac{u_\epsilon^2}{c_\epsilon^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx + \frac{1}{c_\epsilon^2} \int_{\mathbb{R}^2} \frac{u_\epsilon^2}{|x|^{2\beta}} dx \\
&\leq \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx + o_\epsilon(1).
\end{aligned}$$

Thus

$$\limsup_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^2} \leq \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx. \quad (35)$$

Combining (34) and (35), we get the desired result. \square

Corollary 10. *If $\theta < 2$, then $\lambda_\epsilon/c_\epsilon^\theta \rightarrow +\infty$ as $\epsilon \rightarrow 0$.*

Proof. An obvious consequence of Lemma 9. \square

Lemma 11. *For any function $\phi \in C_0^0(\mathbb{R}^2)$, there holds*

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{c_\epsilon u_\epsilon}{\lambda_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} \phi dx = \phi(0).$$

Proof. Let $\phi \in C_0^0(\mathbb{R}^2)$ be fixed. Write for simplicity $h_\epsilon = \lambda_\epsilon^{-1}|x|^{-2\beta}c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}$. Given $0 < \gamma < 1$. Firstly we calculate

$$\int_{u_\epsilon \leq \gamma c_\epsilon} h_\epsilon \phi dx = \frac{c_\epsilon}{\lambda_\epsilon} \int_{u_\epsilon \leq \gamma c_\epsilon} u_\epsilon \phi \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx + \frac{c_\epsilon}{\lambda_\epsilon} \int_{u_\epsilon \leq \gamma c_\epsilon} \frac{u_\epsilon \phi}{|x|^{2\beta}} dx.$$

In view of an obvious analog of (31), there holds

$$\left| \int_{u_\epsilon \leq \gamma c_\epsilon} u_\epsilon \phi \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \right| \leq \left(\sup_{\mathbb{R}^2} |\phi| \right) \int_{\mathbb{R}^2} u_{\epsilon, \gamma} \frac{e^{4\pi(1-\beta-\epsilon)u_{\epsilon, \gamma}^2} - 1}{|x|^{2\beta}} dx = o_\epsilon(1).$$

Note that $u_\epsilon \rightarrow 0$ in $L_{\text{loc}}^q(\mathbb{R}^2)$ for any $q > 0$. We derive

$$\left| \int_{u_\epsilon \leq \gamma c_\epsilon} \frac{u_\epsilon \phi}{|x|^{2\beta}} dx \right| \leq \left(\sup_{\mathbb{R}^2} |\phi| \right) \int_{\text{supp } \phi} \frac{u_\epsilon}{|x|^{2\beta}} dx = o_\epsilon(1).$$

By Corollary 10, $c_\epsilon/\lambda_\epsilon = o_\epsilon(1)$. Therefore

$$\int_{u_\epsilon \leq \gamma c_\epsilon} h_\epsilon \phi dx = o_\epsilon(1). \quad (36)$$

It follows from Lemma 7 that $B_{R_\epsilon^{1/(1-\beta)}} \subset \{u_\epsilon > \gamma c_\epsilon\}$ for sufficiently small $\epsilon > 0$, that

$$\begin{aligned}
\int_{B_{R_\epsilon^{1/(1-\beta)}}} h_\epsilon \phi dx &= \phi(0)(1 + o_\epsilon(1)) \left(\int_{B_R} \frac{e^{8\pi(1-\beta)\varphi}}{|x|^{2\beta}} dx + o_\epsilon(1) \right) \\
&= \phi(0)(1 + o_\epsilon(1) + o_R(1)),
\end{aligned}$$

and that

$$\begin{aligned}
\left| \int_{\{u_\epsilon > \gamma c_\epsilon\} \setminus B_{R_\epsilon^{1/(1-\beta)}}} h_\epsilon \phi dx \right| &\leq \frac{1}{\gamma} \left(\sup_{\mathbb{R}^2} |\phi| \right) \int_{\{u_\epsilon > \gamma c_\epsilon\} \setminus B_{R_\epsilon^{1/(1-\beta)}}} \frac{u_\epsilon^2}{\lambda_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx \\
&\leq \frac{1}{\gamma} \left(\sup_{\mathbb{R}^2} |\phi| \right) \left(1 - \int_{B_R} \frac{e^{8\pi(1-\beta)\varphi}}{|x|^{2\beta}} dx + o_\epsilon(1) \right) \\
&= o_\epsilon(1) + o_R(1).
\end{aligned}$$

It then follows that

$$\lim_{\epsilon \rightarrow 0} \int_{u_\epsilon > \gamma c_\epsilon} h_\epsilon \phi dx = \phi(0). \quad (37)$$

Combining (36) and (37), we complete the proof of the lemma. \square

Lemma 12. $c_\epsilon u_\epsilon \rightarrow G$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\})$ and weakly in $W^{1,q}(\mathbb{R}^2)$ for all $1 < q < 2$, where G is a distributional solution to

$$-\Delta G + \tau G = \delta_0 \quad \text{in } \mathbb{R}^2. \quad (38)$$

Moreover, $G \in W^{1,2}(\mathbb{R}^2 \setminus B_r)$ for any $r > 0$ and G takes the form

$$G(x) = -\frac{1}{2\pi} \log |x| + A_0 + w(x), \quad (39)$$

where A_0 is a constant, $w \in C^1(\mathbb{R}^2)$ and $w(0) = 0$.

Proof. Multiplying both sides of the equation (12) by c_ϵ , one has

$$-\Delta(c_\epsilon u_\epsilon) + \tau(c_\epsilon u_\epsilon) = \frac{c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{\lambda_\epsilon |x|^{2\beta}} \quad \text{in } \mathbb{R}^2. \quad (40)$$

In view of Lemma 11, $h_\epsilon = \lambda_\epsilon^{-1} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}$ is bounded in $L_{\text{loc}}^1(\mathbb{R}^2)$. Using an argument of Li-Ruf ([15], Proposition 3.7), which is adapted from that of Struwe ([24], Theorem 2.2), one concludes that $c_\epsilon u_\epsilon$ is bounded in $W_{\text{loc}}^{1,q}(\mathbb{R}^2)$ for all $1 < q < 2$. Hence $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $W_{\text{loc}}^{1,q}(\mathbb{R}^2)$ for any $1 < q < 2$ and G is a distributional solution to (38). Since $\Delta(G(x) + \frac{1}{2\pi} \log |x|) \in L_{\text{loc}}^p(\mathbb{R}^2)$ for any $p > 2$, (39) follows from elliptic estimates immediately. Applying elliptic estimates to the equation (40), we obtain $c_\epsilon u_\epsilon \rightarrow G$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\})$. Note that $c_\epsilon u_\epsilon \in W^{1,2}(\mathbb{R}^2)$. Multiplying both sides of (40) by $c_\epsilon u_\epsilon$ and integrating by parts on the domain $\mathbb{R}^2 \setminus B_r$ for some $r > 0$, we get

$$\begin{aligned}
\int_{\mathbb{R}^2 \setminus B_r} (|\nabla(c_\epsilon u_\epsilon)|^2 + \tau(c_\epsilon u_\epsilon)^2) dx &= - \int_{\partial B_r} c_\epsilon u_\epsilon \frac{\partial(c_\epsilon u_\epsilon)}{\partial \nu} d\sigma + \int_{\mathbb{R}^2 \setminus B_r} h_\epsilon c_\epsilon u_\epsilon dx \\
&\leq - \int_{\partial B_r} c_\epsilon u_\epsilon \frac{\partial(c_\epsilon u_\epsilon)}{\partial \nu} d\sigma + \frac{c_\epsilon^2 e^{4\pi u_\epsilon^2(r)}}{\lambda_\epsilon} \int_{\mathbb{R}^2 \setminus B_r} \frac{u_\epsilon^2}{|x|^{2\beta}} dx \\
&\leq C_r
\end{aligned}$$

for some constant C_r depending only on r , since $c_\epsilon u_\epsilon \rightarrow G$ in $C_{\text{loc}}^1(\mathbb{R}^2 \setminus \{0\})$. This also leads to

$$\int_{r \leq |x| \leq R} (|\nabla G|^2 + \tau G^2) dx \leq C_r, \quad \forall R > r.$$

Passing to the limit $R \rightarrow \infty$, we have

$$\int_{\mathbb{R}^2 \setminus B_r} (|\nabla G|^2 + \tau G^2) dx \leq C_r.$$

This gives the desired result. \square

2.4. Upper bound estimate

We need a singular version of Carleson-Chang's upper bound estimate, namely Lemma 3.

Lemma 13. *Let $w_\epsilon \in W_0^{1,2}(B_r)$ satisfies $\int_{B_r} |\nabla w_\epsilon|^2 dx \leq 1$, $w_\epsilon \rightharpoonup 0$ weakly in $W_0^{1,2}(B_r)$, and w_ϵ is radially symmetric. Then*

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r} \frac{e^{4\pi(1-\beta)w_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \frac{e\pi}{1-\beta} r^{2(1-\beta)}. \quad (41)$$

Proof. We first prove (41) for $r = 1$.

Denote $w_\epsilon(|x|) = w_\epsilon(x)$. Let $v_\epsilon(x) = \sqrt{1-\beta} w_\epsilon(|x|^{1/(1-\beta)})$. Then

$$\int_{B_1} |\nabla v_\epsilon|^2 dx = \int_{B_1} |\nabla w_\epsilon|^2 dx.$$

Clearly we can assume up to a subsequence, $v_\epsilon \rightharpoonup v_0$ weakly in $W_0^{1,2}(B_1)$, $v_\epsilon \rightarrow v_0$ strongly in $L^2(B_1)$, and $v_\epsilon \rightarrow v_0$ a.e. in B_1 . Also, we can assume $w_\epsilon \rightarrow 0$ a.e. in B_1 . Hence we conclude $v_0 = 0$ a.e. in B_1 . By a change of variable $t = s^{1/(1-\beta)}$, there holds

$$\begin{aligned} \int_{B_1} \frac{e^{4\pi(1-\beta)w_\epsilon^2} - 1}{|x|^{2\beta}} dx &= \int_0^1 \frac{e^{4\pi(1-\beta)w_\epsilon^2(t)} - 1}{t^{2\beta}} 2\pi t dt \\ &= \frac{2\pi}{1-\beta} \int_0^1 s^{(1-2\beta)/(1-\beta)} (e^{4\pi(1-\beta)w_\epsilon^2(s^{1/(1-\beta)})} - 1) s^{\beta/(1-\beta)} ds \\ &= \frac{2\pi}{1-\beta} \int_0^1 s (e^{4\pi v_\epsilon^2(s)} - 1) ds \\ &= \frac{1}{1-\beta} \int_{B_1} (e^{4\pi v_\epsilon^2} - 1) dx. \end{aligned}$$

It follows from Lemma 3 that

$$\limsup_{\epsilon \rightarrow 0} \int_{B_1} \frac{e^{4\pi(1-\beta)w_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \frac{e\pi}{1-\beta}. \quad (42)$$

We next prove (41) for the case of general r . Set $\tilde{w}_\epsilon(x) = w_\epsilon(rx)$ for $x \in B_1$. One can check that

$$\int_{B_1} |\nabla \tilde{w}_\epsilon|^2 dx = \int_{B_r} |\nabla w_\epsilon|^2 dx$$

and that

$$\int_{B_r} \frac{e^{4\pi(1-\beta)w_\epsilon^2} - 1}{|x|^{2\beta}} dx = r^{2(1-\beta)} \int_{B_1} \frac{e^{4\pi(1-\beta)\tilde{w}_\epsilon^2} - 1}{|x|^{2\beta}} dx.$$

This together with (42) gives the desired result. \square

By the equation (12) and $\|u_\epsilon\|_{1,\tau} = 1$, we have

$$\begin{aligned} \int_{B_r} |\nabla u_\epsilon|^2 dx &= 1 - \int_{\mathbb{R}^2 \setminus B_r} (|\nabla u_\epsilon|^2 + \tau u_\epsilon^2) dx - \tau \int_{B_r} u_\epsilon^2 dx \\ &= 1 - \int_{\mathbb{R}^2 \setminus B_r} \frac{u_\epsilon^2}{\lambda_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx + \int_{\partial B_r} u_\epsilon \frac{\partial u_\epsilon}{\partial r} d\sigma - \tau \int_{B_r} u_\epsilon^2 dx. \end{aligned} \quad (43)$$

Since

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B_r} \frac{u_\epsilon^2}{\lambda_\epsilon} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx &= \frac{1}{c_\epsilon^2} \frac{c_\epsilon^2}{\lambda_\epsilon} \int_{\mathbb{R}^2 \setminus B_r} u_\epsilon^2 \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2}}{|x|^{2\beta}} dx \\ &= \frac{o_\epsilon(1)}{c_\epsilon^2}, \end{aligned}$$

$$\int_{\partial B_r} u_\epsilon \frac{\partial u_\epsilon}{\partial r} d\sigma = \frac{1}{c_\epsilon^2} \left(\int_{\partial B_r} G \frac{\partial G}{\partial r} d\sigma + o_\epsilon(1) \right),$$

and

$$\int_{B_r} u_\epsilon^2 dx = \frac{1}{c_\epsilon^2} \left(\int_{B_r} G^2 dx + o_\epsilon(1) \right).$$

Inserting these equations into (43) and noting that $G(x) = -\frac{1}{2\pi} \log |x| + A_0 + w(x)$, we conclude

$$\int_{B_r} |\nabla u_\epsilon|^2 dx = 1 - \frac{1}{c_\epsilon^2} \left(\frac{1}{2\pi} \log \frac{1}{r} + A_0 + o_\epsilon(1) + o_r(1) \right). \quad (44)$$

Denote $s_{\epsilon,r} = \sup_{\partial B_r} u_\epsilon = u_\epsilon(r)$ and $u_{\epsilon,r} = (u_\epsilon - s_{\epsilon,r})^+$, the positive part of $u_\epsilon - s_{\epsilon,r}$. Clearly we have $u_{\epsilon,r} \in W_0^{1,2}(B_r)$. In view of Lemma 13,

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r} \frac{e^{4\pi(1-\beta)u_{\epsilon,r}^2/\tau_{\epsilon,r}} - 1}{|x|^{2\beta}} dx \leq \frac{e\pi}{1-\beta} r^{2(1-\beta)}, \quad (45)$$

where $\tau_{\epsilon,r} = \int_{B_r} |\nabla u_\epsilon|^2 dx$. Moreover, we know from Lemma 7 that $u_\epsilon = c_\epsilon + o_\epsilon(1)$ on $B_{Rr_\epsilon^{1/(1-\beta)}}$. Hence, in view of (44), there holds on $B_{Rr_\epsilon^{1/(1-\beta)}} \subset B_r$,

$$\begin{aligned} 4\pi(1-\beta-\epsilon)u_\epsilon^2 &\leq 4\pi(1-\beta)(u_{\epsilon,r} + s_{\epsilon,r})^2 \\ &= 4\pi(1-\beta)u_{\epsilon,r}^2 + 8\pi(1-\beta)s_{\epsilon,r}u_{\epsilon,r} + o(1) \\ &= 4\pi(1-\beta)u_{\epsilon,r}^2 - 4(1-\beta)\log r + 8\pi(1-\beta)A_0 + o(1) \\ &= 4\pi(1-\beta)u_{\epsilon,r}^2/\tau_{\epsilon,r} - 2(1-\beta)\log r + 4\pi(1-\beta)A_0 + o(1), \end{aligned}$$

where $o(1) \rightarrow 0$ as $\epsilon \rightarrow 0$ first and next $r \rightarrow 0$. Therefore

$$\begin{aligned} \int_{B_{Rr_\epsilon^{1/(1-\beta)}}} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx &\leq r^{-2(1-\beta)} e^{4\pi(1-\beta)A_0 + o(1)} \int_{B_{Rr_\epsilon^{1/(1-\beta)}}} \frac{e^{4\pi(1-\beta)u_{\epsilon,r}^2/\tau_{\epsilon,r}} - 1}{|x|^{2\beta}} dx \\ &= r^{-2(1-\beta)} e^{4\pi(1-\beta)A_0 + o(1)} \int_{B_{Rr_\epsilon^{1/(1-\beta)}}} \frac{e^{4\pi(1-\beta)u_{\epsilon,r}^2/\tau_{\epsilon,r}} - 1}{|x|^{2\beta}} dx + o(1) \\ &\leq r^{-2(1-\beta)} e^{4\pi(1-\beta)A_0 + o(1)} \int_{B_r} \frac{e^{4\pi(1-\beta)u_{\epsilon,r}^2/\tau_{\epsilon,r}} - 1}{|x|^{2\beta}} dx + o(1). \end{aligned} \quad (46)$$

Combining (45) with (46), one concludes for any fixed $R > 0$,

$$\limsup_{\epsilon \rightarrow 0} \int_{B_{Rr_\epsilon^{1/(1-\beta)}}} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \frac{\pi}{1-\beta} e^{1+4\pi(1-\beta)A_0}. \quad (47)$$

In view of Lemma 7, we calculate

$$\begin{aligned} \int_{B_{Rr_\epsilon^{1/(1-\beta)}}} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx &= r_\epsilon^2 \int_{B_R} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2(r_\epsilon^{1/(1-\beta)}y)}{|y|^{2\beta}} dy + o_\epsilon(1) \\ &= \frac{\lambda_\epsilon}{c_\epsilon^2} \left(\int_{B_R} \frac{e^{8\pi(1-\beta)\varphi(y)}}{|y|^{2\beta}} dy + o_\epsilon(1) \right) + o_\epsilon(1) \\ &= \frac{\lambda_\epsilon}{c_\epsilon^2} (1 + o_R(1) + o_\epsilon(1)) + o_\epsilon(1). \end{aligned}$$

As a consequence,

$$\lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_{B_{Rr_\epsilon^{1/(1-\beta)}}} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx = \lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \quad (48)$$

Combining (47), (48) and (28), in view of (13) and (14), we arrive at

$$\sup_{u \in W^{1,2}(\mathbb{R}^2), \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)u^2} - 1}{|x|^{2\beta}} dx = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta-\epsilon)u_\epsilon^2} - 1}{|x|^{2\beta}} dx \leq \frac{\pi}{1-\beta} e^{1+4\pi(1-\beta)A_0}. \quad (49)$$

2.5. Test function computation

We now construct test functions such that (49) does not hold. Precisely we construct a sequence of functions $\phi_\epsilon \in W^{1,2}(\mathbb{R}^2)$ satisfying $\|\phi_\epsilon\|_{1,\tau} = 1$ and

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)\phi_\epsilon^2} - 1}{|x|^{2\beta}} dx > \frac{\pi}{1-\beta} e^{1+4\pi(1-\beta)A_0} \quad (50)$$

for sufficiently small $\epsilon > 0$. For this purpose we set

$$\phi_\epsilon(x) = \begin{cases} c + \frac{1}{c} \left(-\frac{1}{4\pi(1-\beta)} \log \left(1 + \frac{\pi}{1-\beta} \frac{|x|^{2(1-\beta)}}{\epsilon^{2(1-\beta)}} \right) + b \right), & x \in \overline{B_{R\epsilon}} \\ \frac{G}{c}, & x \in \mathbb{R}^2 \setminus B_{R\epsilon}, \end{cases}$$

where G is given as in Lemma 12, $R = (-\log \epsilon)^{1/(1-\beta)}$, b and c are constants depending only on ϵ to be determined later. To ensure $\phi_\epsilon \in W^{1,2}(\mathbb{R}^2)$, we let

$$c + \frac{1}{c} \left(-\frac{1}{4\pi(1-\beta)} \log \left(1 + \frac{\pi}{1-\beta} R^{2(1-\beta)} \right) + b \right) = \frac{1}{c} \left(-\frac{1}{2\pi} \log(R\epsilon) + A_0 + w(R\epsilon) \right).$$

This leads to

$$c^2 = \frac{1}{4\pi(1-\beta)} \log \frac{\pi}{1-\beta} + A_0 - b - \frac{1}{2\pi} \log \epsilon + O\left(\frac{1}{R^{2-2\beta}}\right) + O(R\epsilon). \quad (51)$$

Now we calculate

$$\begin{aligned}
\int_{\mathbb{R}^2 \setminus B_{R\epsilon}} (|\nabla \phi_\epsilon|^2 + \tau \phi_\epsilon^2) dx &= \frac{1}{c^2} \int_{\mathbb{R}^2 \setminus B_{R\epsilon}} (|\nabla G|^2 + \tau G^2) dx \\
&= -\frac{1}{c^2} \int_{\partial B_{R\epsilon}} G \frac{\partial G}{\partial r} d\sigma \\
&= \frac{1}{c^2} \left(-\frac{1}{2\pi} \log(R\epsilon) + A_0 + O(R\epsilon \log(R\epsilon)) \right)
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
\int_{B_{R\epsilon}} |\nabla \phi_\epsilon|^2 dx &= \frac{1}{4\pi c^2} \int_0^{R\epsilon} \frac{2r^{3-4\beta}}{(r^{2(1-\beta)} + \frac{1-\beta}{\pi} \epsilon^{2(1-\beta)})^2} dr \\
&= \frac{1}{4\pi(1-\beta)c^2} \left(\log\left(1 + \frac{\pi}{1-\beta} R^{2-2\beta}\right) + \frac{1}{1 + \frac{\pi}{1-\beta} R^{2-2\beta}} - 1 \right) \\
&= \frac{1}{4\pi(1-\beta)c^2} \left(\log \frac{\pi}{1-\beta} + \log R^{2-2\beta} - 1 + O\left(\frac{1}{R^{2-2\beta}}\right) \right).
\end{aligned} \tag{53}$$

Moreover, we require b to be bounded with respect to ϵ . It then follows that

$$\int_{B_{R\epsilon}} \phi_\epsilon^2 dx = \frac{1}{c^2} \int_{B_{R\epsilon}} \left(c^2 - \frac{1}{4\pi(1-\beta)} \log\left(1 + \frac{\pi}{1-\beta} \frac{|x|^{2(1-\beta)}}{\epsilon^{2(1-\beta)}}\right) + b \right)^2 dx = O(R\epsilon). \tag{54}$$

Combining (52), (53) and (54), we obtain

$$\|\phi_\epsilon\|_{1,\tau}^2 = \frac{1}{c^2} \left(-\frac{1}{2\pi} \log \epsilon + A_0 - \frac{1}{4\pi(1-\beta)} + \frac{1}{4\pi(1-\beta)} \log \frac{\pi}{1-\beta} + O\left(\frac{1}{R^{2-2\beta}}\right) \right).$$

Setting $\|\phi_\epsilon\|_{1,\tau} = 1$, we have

$$c^2 = -\frac{1}{2\pi} \log \epsilon + A_0 - \frac{1}{4\pi(1-\beta)} + \frac{1}{4\pi(1-\beta)} \log \frac{\pi}{1-\beta} + O\left(\frac{1}{R^{2-2\beta}}\right), \tag{55}$$

which together with (51) leads to

$$b = \frac{1}{4\pi(1-\beta)} + O\left(\frac{1}{R^{2-2\beta}}\right). \tag{56}$$

For all $x \in B_{R\epsilon}$, it follows from (55) and (56) that

$$\begin{aligned}
4\pi(1-\beta)\phi_\epsilon^2(x) &\geq 4\pi(1-\beta)c^2 + 8\pi(1-\beta)b - 2 \log \left(1 + \frac{\pi}{1-\beta} \frac{|x|^{2-2\beta}}{\epsilon^{2-2\beta}} \right) \\
&= -2 \log \left(1 + \frac{\pi}{1-\beta} \frac{|x|^{2-2\beta}}{\epsilon^{2-2\beta}} \right) - 2(1-\beta) \log \epsilon \\
&\quad + 4\pi(1-\beta)A_0 + \log \frac{\pi}{1-\beta} + 1 + O\left(\frac{1}{R^{2-2\beta}}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{B_{R\epsilon}} \frac{e^{4\pi(1-\beta)\phi_\epsilon^2} - 1}{|x|^{2\beta}} dx &\geq \frac{\pi}{1-\beta} \epsilon^{-2(1-\beta)} e^{1+4\pi(1-\beta)A_0 + O(\frac{1}{R^{2-2\beta}})} \\
&\times \int_{B_{R\epsilon}} \frac{1}{(1 + \frac{\pi}{1-\beta} \frac{|x|^{2(1-\beta)}}{\epsilon^{2(1-\beta)}})^2 |x|^{2\beta}} dx + O((R\epsilon)^{2-2\beta}) \\
&= \frac{\pi}{1-\beta} e^{1+4\pi(1-\beta)A_0 + O(\frac{1}{R^{2-2\beta}})} \\
&\times \int_{B_R} \frac{1}{(1 + \frac{\pi}{1-\beta} |y|^{2(1-\beta)})^2 |y|^{2\beta}} dy + O((R\epsilon)^{2-2\beta}) \\
&= \frac{\pi}{1-\beta} e^{1+4\pi(1-\beta)A_0} + O(\frac{1}{R^{2-2\beta}}). \tag{57}
\end{aligned}$$

Also we calculate

$$\int_{\mathbb{R}^2 \setminus B_{R\epsilon}} \frac{e^{4\pi(1-\beta)\phi_\epsilon^2} - 1}{|x|^{2\beta}} dx \geq \frac{4\pi(1-\beta)}{c^2} \int_{\mathbb{R}^2 \setminus B_{R\epsilon}} \frac{G^2}{|x|^{2\beta}} dx = \frac{4\pi(1-\beta)}{c^2} \left(\int_{\mathbb{R}^2} \frac{G^2}{|x|^{2\beta}} dx + o_\epsilon(1) \right). \tag{58}$$

Combining (57) and (58) and noting that $c^2/R^{2-2\beta} = o_\epsilon(1)$, we have

$$\int_{\mathbb{R}^2} \frac{e^{4\pi(1-\beta)\phi_\epsilon^2} - 1}{|x|^{2\beta}} dx \geq \frac{\pi}{1-\beta} e^{1+4\pi(1-\beta)A_0} + \frac{4\pi(1-\beta)}{c^2} \left(\int_{\mathbb{R}^2} \frac{G^2}{|x|^{2\beta}} dx + o_\epsilon(1) \right).$$

Therefore we conclude (50) for sufficiently small $\epsilon > 0$.

2.6. Completion of the proof of Theorem 2

Comparing (50) with (49), we conclude that c_ϵ must be bounded. Then applying elliptic estimates to (12), we get the desired extremal function. This ends the proof of Theorem 2. \square

3. N -dimensional case

In this section, we prove Theorems 1 and 2 in the case that $N \geq 3$. We put emphasis on the essential difference between 2 dimensions and N dimensions. In the sequel, we denote $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2} \nabla u)$ for any $u \in W^{1,N}(\mathbb{R}^N)$ ($N \geq 3$). Let $\zeta : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined as in (6). Obviously, one has

$$\frac{d}{dt} \zeta(N, t) = \zeta(N-1, t). \tag{59}$$

In view of ([27], Lemma 2.1), there holds for all $p \geq 1$ and $t \geq 0$,

$$(\zeta(N, t))^p \leq \zeta(N, pt). \tag{60}$$

3.1. A priori estimates

We need elliptic estimates for quasi-linear equations as below.

Theorem 14. *Let $R > 0$ be fixed. Suppose that $u \in W^{1,N}(B_R)$ is a weak solution of*

$$-\Delta_N u = f \quad \text{in } B_R \subset \mathbb{R}^N.$$

Then the following a priori estimates hold:

- **(Harnack inequality)** If $u \geq 0$ and $f \in L^p(B_R)$ for some $p > 1$, then there exists some constant C depending only on N, R and p such that $\sup_{B_{R/2}} u \leq C(\inf_{B_{R/2}} u + \|f\|_{L^p(B_R)})$;
- **(C^α -estimate)** If $\|u\|_{L^\infty(B_R)} \leq L$ and $\|f\|_{L^p(B_R)} \leq M$ for some $p > 1$, then there exists two constants $0 < \alpha \leq 1$ and C depending only on N, R, p, L and M such that $u \in C^\alpha(\overline{B}_{R/2})$ and $\|u\|_{C^\alpha(\overline{B}_{R/2})} \leq C$;
- **($C^{1,\alpha}$ -estimate)** If $\|u\|_{L^\infty(B_R)} \leq L$ and $\|f\|_{L^\infty(B_R)} \leq M$, then there exists two constants $0 < \alpha \leq 1$ and C depending only on N, R, L and M such that $u \in C^{1,\alpha}(\overline{B}_{R/2})$ and $\|u\|_{C^{1,\alpha}(\overline{B}_{R/2})} \leq C$.

In the above theorem, the first two estimates were obtained by J. Serrin ([22], Theorems 6 and 8), while the third estimate was proved by Tolksdorf ([25], Theorem 1).

3.2. Extremal functions for subcritical Trudinger-Moser inequalities

In this subsection, we prove Theorem 1 in the case $N \geq 3$. The proof is based on a direct method of variation. Throughout this section, we denote for simplicity

$$\beta_{N,\epsilon} = \alpha_N(1 - \beta - \epsilon). \quad (61)$$

Proof of Theorem 1. Let \mathcal{S}_N be a subset of $W^{1,N}(\mathbb{R}^N)$ consisting of all functions, which are nonnegative decreasing radially symmetric almost everywhere. By a rearrangement argument, we have

$$\Lambda_{N,\beta,\tau,\epsilon} = \sup_{u \in \mathcal{S}_N, \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} u^{\frac{N}{N-1}})}{|x|^{N\beta}} dx,$$

where $\Lambda_{N,\beta,\tau,\epsilon}$ is defined as in (7) and $\beta_{N,\epsilon}$ is defined as in (61). Take $u_j \in \mathcal{S}_N$ with $\|u_j\|_{1,\tau} \leq 1$ such that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} u_j^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \Lambda_{N,\beta,\tau,\epsilon}.$$

Up to a subsequence, we can find some function u_ϵ such that u_j converges to u_ϵ weakly in $W^{1,N}(\mathbb{R}^N)$, strongly in $L^q_{\text{loc}}(\mathbb{R}^N)$ for any $q > 0$, and a.e. in \mathbb{R}^N . Obviously $u_\epsilon \in \mathcal{S}_N$. It follows from the weak convergence of u_j in $W^{1,N}(\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N} |\nabla u_\epsilon|^N dx = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_\epsilon|^{N-2} \nabla u_j \nabla u_\epsilon dx,$$

which together with the Hölder inequality leads to

$$\int_{\mathbb{R}^N} |\nabla u_\epsilon|^N dx \leq \limsup_{j \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_j|^N dx. \quad (62)$$

While it follows from $u_j \rightarrow u_\epsilon$ in $L^q_{\text{loc}}(\mathbb{R}^N)$ for any $q > 0$ that for any fixed $R > 0$,

$$\int_{B_R} u_\epsilon^N dx = \lim_{j \rightarrow \infty} \int_{B_R} u_j^N dx. \quad (63)$$

Combining (62) and (63), one can easily see that $\|u_\epsilon\|_{1,\tau} \leq \limsup_{j \rightarrow \infty} \|u_j\|_{1,\tau} \leq 1$. Given any $\nu > 0$, there hold

$$\int_{|x| > \nu^{-\frac{1}{N\beta}}} \frac{\zeta(N, \beta_{N,\epsilon} u_j^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \leq \nu \int_{\mathbb{R}^N} \zeta(N, \beta_{N,\epsilon} u_j^{\frac{N}{N-1}}) dx \leq \nu \Lambda_{N,0,\tau}, \quad (64)$$

$$\int_{|x| > \nu^{-\frac{1}{N\beta}}} \frac{\zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \leq \nu \int_{\mathbb{R}^N} \zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}) dx \leq \nu \Lambda_{N,0,\tau}, \quad (65)$$

where $\Lambda_{N,0,\tau}$ is defined as in (8). In view of (59), we have by the mean value theorem,

$$\begin{aligned} \zeta(N, \beta_{N,\epsilon} u_j^{\frac{N}{N-1}}) - \zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}) &= \zeta(N-1, \vartheta) \beta_{N,\epsilon} (u_j^{\frac{N}{N-1}} - u_\epsilon^{\frac{N}{N-1}}) \\ &\leq \max\{\zeta(N-1, \beta_{N,\epsilon} u_j^{\frac{N}{N-1}}), \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})\} \\ &\quad \times \beta_{N,\epsilon} (u_j^{\frac{N}{N-1}} - u_\epsilon^{\frac{N}{N-1}}), \end{aligned} \quad (66)$$

where ϑ lies between $\beta_{N,\epsilon} u_j^{\frac{N}{N-1}}$ and $\beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}$. Employing (60) and (4), we calculate for some $p, 1 < p < \min\{\frac{1}{1-\epsilon}, \frac{1}{\beta}\}$,

$$\begin{aligned} \int_{|x| \leq \nu^{-\frac{1}{N\beta}}} \frac{(\zeta(N-1, \vartheta))^p}{|x|^{N\beta p}} dx &\leq \int_{|x| \leq \nu^{-\frac{1}{N\beta}}} \frac{\zeta(N-1, p\vartheta)}{|x|^{N\beta p}} dx \\ &= \int_{|x| \leq \nu^{-\frac{1}{N\beta}}} \frac{\zeta(N, p\vartheta)}{|x|^{N\beta p}} dx + \int_{|x| \leq \nu^{-\frac{1}{N\beta}}} \frac{1}{(N-2)!} \frac{(p\vartheta)^{N-2}}{|x|^{N\beta p}} dx \\ &\leq \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} p u_j^{\frac{N}{N-1}})}{|x|^{N\beta p}} dx + \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} p u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta p}} dx + C_1 \leq C, \end{aligned}$$

where C_1 is a constant depending only on N, β, p , while C is a constant depending on N, β, ϵ and p . This together with (64)-(66), the Hölder inequality and the fact that $u_j \rightarrow u_\epsilon$ in $L_{\text{loc}}^q(\mathbb{R}^N)$ for any $q > 0$ implies that

$$\Lambda_{N,\beta,\tau,\epsilon} = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} u_j^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx.$$

Clearly we must have $\|u_\epsilon\|_{1,\tau} = 1$. Moreover, by a straightforward calculation, we derive the Euler-Lagrange equation of u_ϵ as follows:

$$\begin{cases} -\Delta_N u_\epsilon + \tau u_\epsilon^{N-1} = \frac{1}{\lambda_\epsilon} \frac{u_\epsilon^{1/(N-1)}}{|x|^{N\beta}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}) & \text{in } \mathbb{R}^N, \\ \lambda_\epsilon = \int_{\mathbb{R}^N} |x|^{-N\beta} u_\epsilon^{\frac{N}{N-1}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}) dx. \end{cases} \quad (67)$$

Applying Theorem 14 to (67), we have $u_\epsilon \in C^1(\mathbb{R}^N \setminus \{0\}) \cap C^0(\mathbb{R}^N)$. This completes the proof of the theorem. \square

The remaining part of this section is devoted to the proof of Theorem 2.

3.3. Elementary properties of u_ϵ

Similar to Lemma 5, we have the following:

Lemma 15. *Let λ_ϵ be defined as in (67). Then there holds $\liminf_{\epsilon \rightarrow 0} \lambda_\epsilon > 0$.*

Proof. Employing the Lebesgue dominated convergence theorem and noting that u_ϵ is a maximizer for subcritical Trudinger-Moser inequalities, we have for all $u \in W^{1,N}(\mathbb{R}^N)$ with $\|u\|_{1,\tau} \leq 1$,

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)|u|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon}|u|^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \\ &\leq \limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon}u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx. \end{aligned}$$

One easily concludes

$$\Lambda_{N,\beta,\tau} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon}u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx. \quad (68)$$

Since for any $t \geq 0$,

$$t\zeta(N-1, t) = \sum_{k=N-2}^{\infty} \frac{t^{k+1}}{k!} = \sum_{k=N-1}^{\infty} \frac{t^k}{(k-1)!} \geq \sum_{k=N-1}^{\infty} \frac{t^k}{k!} = \zeta(N, t),$$

one has

$$\lambda_\epsilon \geq \frac{1}{\beta_{N,\epsilon}} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon}u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \frac{1}{\alpha_N(1-\beta)} \Lambda_{N,\beta,\tau} + o_\epsilon(1).$$

Thus we get the desired result since $\Lambda_{N,\beta,\tau} > 0$. \square

Since $\|u_\epsilon\|_{1,\tau} = 1$, one can find some function u_0 such that u_ϵ converges to u_0 weakly in $W^{1,N}(\mathbb{R}^N)$, strongly in $L_{\text{loc}}^q(\mathbb{R}^N)$ for any $q > 0$, and a.e. in \mathbb{R}^N . Denote $c_\epsilon = u_\epsilon(0)$. If c_ϵ is a bounded sequence, then applying a priori estimates in Theorem 14 to (67), we conclude that $u_\epsilon \rightarrow u_0$ in $C_{\text{loc}}^0(\mathbb{R}^N) \cap C_{\text{loc}}^1(\mathbb{R}^N \setminus \{0\})$. It is not difficult to see that

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)u_0^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \Lambda_{N,\beta,\tau}.$$

This also implies that $\|u_0\|_{1,\tau} = 1$ and thus u_0 is the desired maximizer for the critical Trudinger-Moser functional. In the following, without loss of generality, we assume $c_\epsilon \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

Lemma 16. $u_0 \equiv 0$ and up to a subsequence, $|\nabla u_\epsilon|^N dx \rightharpoonup \delta_0$ weakly in the sense of measure.

Proof. We first prove that $|\nabla u_\epsilon|^N dx \rightharpoonup \delta_0$. Suppose not. There exists $r_0 > 0$ such that

$$\limsup_{\epsilon \rightarrow 0} \int_{B_{r_0}} |\nabla u_\epsilon|^N dx \leq \eta < 1.$$

Note that u_ϵ is decreasing radially symmetric. Let $\tilde{u}_\epsilon(x) = u_\epsilon(x) - u_\epsilon(r_0)$ for $x \in B_{r_0}$. Then $\tilde{u}_\epsilon \in W_0^{1,N}(B_{r_0})$ satisfies $\|\nabla \tilde{u}_\epsilon\|_{L^N(B_{r_0})} \leq \eta < 1$. Denote

$$f_\epsilon(x) = \frac{1}{\lambda_\epsilon} \frac{u_\epsilon^{1/(N-1)}(x)}{|x|^{N\beta}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}(x)).$$

There holds for any $p > 1$, $p_1 > 1$ and $1/p_1 + 1/p_2 = 1$,

$$\begin{aligned} \int_{B_{r_0}} f_\epsilon^p(x) dx &\leq \int_{B_{r_0}} \frac{1}{\lambda_\epsilon^p} \frac{u_\epsilon^{p/(N-1)}}{|x|^{N\beta p}} \zeta(N-1, \beta_{N,\epsilon} p u_\epsilon^{\frac{N}{N-1}}(x)) dx \\ &\leq \frac{1}{\lambda_\epsilon^p} \left(\int_{B_{r_0}} \frac{u_\epsilon^{pp_1/(N-1)}}{|x|^{N\beta p}} dx \right)^{1/p_1} \left(\int_{B_{r_0}} \frac{\zeta(N-1, \beta_{N,\epsilon} p p_2 u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta p}} dx \right)^{1/p_2} \\ &\leq \frac{1}{\lambda_\epsilon^p} \left(\int_{B_{r_0}} \frac{u_\epsilon^{pp_1/(N-1)}}{|x|^{N\beta p}} dx \right)^{1/p_1} \left(\int_{B_{r_0}} \frac{e^{\alpha_N(1-\beta) p p_2 u_\epsilon^{\frac{N}{N-1}}}}{|x|^{N\beta p}} dx \right)^{1/p_2}. \end{aligned} \quad (69)$$

Since u_ϵ is nonnegative decreasing radially symmetric, one has $\int_{B_{r_0}} u_\epsilon^N dx \geq u_\epsilon^N(r_0) \frac{\omega_{N-1}}{N} r_0^N$. It follows that

$$u_\epsilon(r_0) \leq \left(\frac{N}{\omega_{N-1}} \right)^{1/N} \frac{\|u_\epsilon\|_{L^N(B_{r_0})}}{r_0} \leq \left(\frac{N}{\omega_{N-1} \tau} \right)^{1/N} \frac{1}{r_0}. \quad (70)$$

Here we have used $\|u_\epsilon\|_{1,\tau} = 1$. For any $\nu > 0$, there exists some constant C_0 depending only on N and ν such that for all $x \in B_{r_0}$,

$$u_\epsilon^{\frac{N}{N-1}}(x) \leq (1 + \nu) \tilde{u}_\epsilon^{\frac{N}{N-1}}(x) + C_0 u_\epsilon^{\frac{N}{N-1}}(r_0). \quad (71)$$

Choosing $p > 1$, $p_2 > 1$ sufficiently close to 1 and $\nu > 0$ sufficiently small such that $(1-\beta) p p_2 (1+\nu) + \beta p < 1$, inserting (70) and (71) into (69), and noting that u_ϵ is bounded in $L^q(B_{r_0})$ for any fixed $q > 0$, one can see from (3) and Lemma 15 that f_ϵ is bounded in $L^p(B_{r_0})$. By the elliptic estimate (Theorem 14), u_ϵ is uniformly bounded in $B_{r_0/2}$ contradicting $c_\epsilon \rightarrow +\infty$. This confirms that $|\nabla u_\epsilon|^N dx \rightharpoonup \delta_0$ in the sense of measure.

Next we prove $u_0 \equiv 0$. It follows from $\|u_\epsilon\|_{1,\tau} = 1$ and $|\nabla u_\epsilon|^N dx \rightharpoonup \delta_0$ that $\|u_\epsilon\|_{L^N(\mathbb{R}^N)} = o_\epsilon(1)$, which leads to

$$\int_{\mathbb{R}^N} u_0^N dx \leq \limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} u_\epsilon^N dx = 0.$$

Therefore $u_0 \equiv 0$ and the proof of the lemma is completed. \square

3.4. Blow-up analysis

Let

$$r_\epsilon = \lambda_\epsilon^{1/N} c_\epsilon^{-1/(N-1)} e^{-\beta_{N,\epsilon} c_\epsilon^{N/(N-1)/N}}. \quad (72)$$

Define

$$\psi_{N,\epsilon}(x) = c_\epsilon^{-1} u_\epsilon(r_\epsilon^{1/(1-\beta)} x) \quad (73)$$

and

$$\varphi_{N,\epsilon}(x) = c_\epsilon^{1/(N-1)} (u_\epsilon(r_\epsilon^{1/(1-\beta)} x) - c_\epsilon). \quad (74)$$

Analogous to Lemma 7, we have the following:

Lemma 17. Let r_ϵ , $\psi_{N,\epsilon}$ and $\varphi_{N,\epsilon}$ be defined as in (72)-(74). Then (i) for any $\gamma < \alpha_N(1-\beta)/N$, there holds $r_\epsilon e^{\gamma c_\epsilon^{N/(N-1)}} \rightarrow 0$ as $\epsilon \rightarrow 0$; (ii) $\psi_{N,\epsilon} \rightarrow 1$ in $C_{\text{loc}}^1(\mathbb{R}^N \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^N)$; (iii) $\varphi_{N,\epsilon} \rightarrow \varphi_N$ in $C_{\text{loc}}^1(\mathbb{R}^N \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^N)$, where

$$\varphi_N(x) = -\frac{N-1}{\alpha_N(1-\beta)} \log \left(1 + \frac{\alpha_N}{N^{N/(N-1)}(1-\beta)^{1/(N-1)}} |x|^{\frac{N}{N-1}(1-\beta)} \right).$$

Moreover

$$\int_{\mathbb{R}^N} \frac{e^{\frac{N}{N-1}\alpha_N(1-\beta)\varphi_N}}{|x|^{N\beta}} dx = 1. \quad (75)$$

Proof. (i) In view of (72), one has

$$\begin{aligned} r_\epsilon^N e^{N\gamma c_\epsilon^{\frac{N}{N-1}}} &= c_\epsilon^{-\frac{N}{N-1}} e^{-\alpha_N(1-\beta-\epsilon-\frac{N\gamma}{\alpha_N})c_\epsilon^{\frac{N}{N-1}}} \int_{\mathbb{R}^N} \frac{u_\epsilon^{\frac{N}{N-1}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \\ &\leq c_\epsilon^{-\frac{N}{N-1}} e^{-\alpha_N(1-\beta-\epsilon-\frac{N\gamma}{\alpha_N})c_\epsilon^{\frac{N}{N-1}}} \int_{\mathbb{R}^N} \frac{u_\epsilon^{\frac{N}{N-1}} e^{\beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx \\ &\leq c_\epsilon^{-\frac{N}{N-1}} \int_{\mathbb{R}^N} \frac{u_\epsilon^{\frac{N}{N-1}} e^{N\gamma u_\epsilon^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx \end{aligned} \quad (76)$$

Since $N\gamma < \alpha_N(1-\beta)$, one can see from (3) that

$$\int_{\mathbb{R}^N} \frac{u_\epsilon^{\frac{N}{N-1}} e^{N\gamma u_\epsilon^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx \leq C$$

for some constant C , which together with (76) implies that $r_\epsilon e^{\gamma c_\epsilon^{N/(N-1)}} = o_\epsilon(1)$.

(ii) Clearly $\psi_{N,\epsilon}$ is a distributional solution to

$$-\Delta_N \psi_{N,\epsilon}(x) = -\tau r_\epsilon^{\frac{1}{1-\beta}} \psi_{N,\epsilon}^{N-1}(x) + c_\epsilon^{-N} |x|^{-N\beta} \psi_{N,\epsilon}^{\frac{1}{N-1}}(x) e^{-\beta_{N,\epsilon} c_\epsilon^{N/(N-1)}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}(r_\epsilon^{\frac{1}{1-\beta}} x)). \quad (77)$$

Applying Theorem 14 to (77), we have $\psi_{N,\epsilon} \rightarrow \psi_N$ in $C_{\text{loc}}^1(\mathbb{R}^N \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^N)$, where ψ_N is a distributional solution to $\Delta_N \psi_N = 0$ in \mathbb{R}^N . Clearly $\psi_N \equiv 1$ on \mathbb{R}^N .

(iii) In view of (67), we derive the equation of $\varphi_{N,\epsilon}$ as follows.

$$\begin{aligned} -\Delta_N \varphi_{N,\epsilon}(x) &= g_{N,\epsilon}(x) \\ &= |x|^{-N\beta} \psi_{N,\epsilon}^{\frac{1}{N-1}}(x) e^{-\beta_{N,\epsilon} c_\epsilon^{N/(N-1)}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}(r_\epsilon^{\frac{1}{1-\beta}} x)) - \tau r_\epsilon^{\frac{1}{1-\beta}} c_\epsilon^N \psi_{N,\epsilon}^{N-1}(x). \end{aligned} \quad (78)$$

Let R, r be any two positive numbers such that $R > 4r$. Clearly $g_{N,\epsilon}$ is bounded in $L^p(B_R)$ for some $p > 1$. Moreover, $-\varphi_{N,\epsilon} \geq 0$. Theorem 14 implies that $\varphi_{N,\epsilon}$ is uniformly bounded in $B_{R/2}$. While $g_{N,\epsilon}$ is bounded in $L^\infty(B_R \setminus B_r)$. Hence we have by applying Theorem 14 to (78), $\varphi_{N,\epsilon}$ is bounded in $C^{1,\alpha}(B_{R/2} \setminus B_{2r})$ for some $0 < \alpha < 1$. Therefore up to a subsequence, there exists some function φ_N such that $\varphi_{N,\epsilon} \rightarrow \varphi_N$ in $C_{\text{loc}}^1(\mathbb{R}^N \setminus \{0\}) \cap C_{\text{loc}}^0(\mathbb{R}^N)$. To derive the equation of φ_N , we estimate

$$0 \leq e^{-\beta_{N,\epsilon} c_\epsilon^{N/(N-1)}} \sum_{k=0}^{N-3} \frac{\beta_{N,\epsilon}^k u_\epsilon^{\frac{Nk}{N-1}}(r_\epsilon^{\frac{1}{1-\beta}} x)}{k!} \leq e^{-\beta_{N,\epsilon} c_\epsilon^{N/(N-1)}} \sum_{k=0}^{N-3} \frac{\beta_{N,\epsilon}^k c_\epsilon^{\frac{Nk}{N-1}}}{k!} = o_\epsilon(1)$$

uniformly on B_R for any $R > 0$. Moreover, by the mean value theorem, we have

$$\begin{aligned} u_\epsilon^{\frac{N}{N-1}}(r_\epsilon^{\frac{1}{1-\beta}}x) - c_\epsilon^{\frac{N}{N-1}} &= \frac{N}{N-1} \xi_\epsilon^{\frac{1}{N-1}} (u_\epsilon(r_\epsilon^{\frac{1}{1-\beta}}x) - c_\epsilon) \\ &= \frac{N}{N-1} (\xi_\epsilon/c_\epsilon)^{\frac{1}{N-1}} \varphi_{N,\epsilon}(x) \\ &= \frac{N}{N-1} \varphi_N(x) + o_\epsilon(1), \end{aligned} \quad (79)$$

where ξ_ϵ lies between $u_\epsilon(r_\epsilon^{1/(1-\beta)}x)$ and c_ϵ , and $o_\epsilon(1) \rightarrow 0$ uniformly on B_R for any fixed $R > 0$. Hence

$$e^{-\beta_{N,\epsilon} c_\epsilon^{N/(N-1)}} \zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}(r_\epsilon^{\frac{1}{1-\beta}}x)) = e^{\alpha_N(1-\beta)\frac{N}{N-1}\varphi_N(x)} + o_\epsilon(1).$$

Furthermore, we obtain the equation of φ_N as follows:

$$\begin{cases} -\Delta_N \varphi_N(x) = \frac{e^{\alpha_N(1-\beta)\frac{N}{N-1}\varphi_N(x)}}{|x|^{N\beta}} & \text{in } \mathbb{R}^N, \\ \varphi_N(0) = \max_{\mathbb{R}^N} \varphi_N = 0. \end{cases} \quad (80)$$

Since $\varphi_{N,\epsilon}$ is decreasingly symmetric on \mathbb{R}^N , φ_N is also decreasingly symmetric. Denote $\varphi_N(r) = \varphi_N(x)$, where $r = |x|$ and $x \in \mathbb{R}^N$. Then (80) can be reduced to an ordinary differential equation, namely

$$\begin{cases} ((-r\varphi'_N(r))^{N-1})' = r^{N-1-N\beta} e^{\alpha_N(1-\beta)\frac{N}{N-1}\varphi_N(r)} \\ \varphi_N(0) = 0. \end{cases} \quad (81)$$

By a standard uniqueness result of ordinary differential equations (see for example [15]), we can solve (81) as

$$\varphi_N(r) = -\frac{N-1}{\alpha_N(1-\beta)} \log(1 + c_N r^{\frac{N}{N-1}(1-\beta)}),$$

where $c_N = \alpha_N N^{-N/(N-1)}(1-\beta)^{-1/(N-1)}$. It then follows that

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{e^{\alpha_N(1-\beta)\frac{N}{N-1}\varphi_N(x)}}{|x|^{N\beta}} dx &= \omega_{N-1} \int_0^\infty \frac{r^{N-1-N\beta}}{(1 + c_N r^{\frac{N}{N-1}(1-\beta)})^N} dr \\ &= \omega_{N-1} \frac{N-1}{N(1-\beta)} \int_0^\infty \frac{t^{N-2}}{(1 + c_N t)^N} dt. \end{aligned} \quad (82)$$

Integration by parts gives

$$\begin{aligned} I_N &= (N-1) \int_0^\infty \frac{t^{N-2}}{(1 + c_N t)^N} dt \\ &= -\frac{1}{c_N} \int_0^\infty t^{N-2} d(1 + c_N t)^{1-N} \\ &= -\frac{1}{c_N} t^{N-2} (1 + c_N t)^{1-N} \Big|_0^\infty + \frac{N-2}{c_N} \int_0^\infty t^{N-3} (1 + c_N t)^{1-N} dt \\ &= \frac{1}{c_N} I_{N-1}. \end{aligned}$$

Iteration leads to

$$I_N = \frac{1}{c_N^{N-2}} I_2 = \frac{1}{c_N^{N-2}} \int_0^\infty \frac{1}{(1 + c_N t)^2} dt = \frac{1}{c_N^{N-1}}. \quad (83)$$

Inserting (83) into (82), we obtain

$$\int_{\mathbb{R}^N} \frac{e^{\frac{N}{N-1} \alpha_N (1-\beta) \varphi_N(x)}}{|x|^{N\beta}} dx = \frac{\omega_{N-1}}{N(1-\beta)} \frac{1}{c_N^{N-1}} = 1.$$

This completes the proof of the lemma. \square

For any $0 < \gamma < 1$, we set $u_{\epsilon, \gamma} = \min\{u_\epsilon, \gamma c_\epsilon\}$. Then we have the following:

Lemma 18. *For any $0 < \gamma < 1$, there holds $\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_{\epsilon, \gamma}|^N dx = \gamma$.*

Proof. Testing the equation (67) by $u_{\epsilon, \gamma}$, we have for any fixed $R > 0$,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_{\epsilon, \gamma}|^N dx &= -\tau \int_{\mathbb{R}^N} u_\epsilon^{N-1} u_{\epsilon, \gamma} dx + \frac{1}{\lambda_\epsilon} \int_{\mathbb{R}^N} u_{\epsilon, \gamma} \frac{u_\epsilon^{1/(N-1)}}{|x|^{N\beta}} \zeta(N-1, \beta_{N, \epsilon} u_\epsilon^{N/(N-1)}) dx \\ &\geq \frac{1}{\lambda_\epsilon} \int_{B_{R_\epsilon^{1/(1-\beta)}}} \gamma c_\epsilon \frac{u_\epsilon^{1/(N-1)}}{|x|^{N\beta}} e^{\beta_{N, \epsilon} u_\epsilon^{N/(N-1)}} dx + o_\epsilon(1) \\ &= (1 + o_\epsilon(1)) \gamma \int_{B_R} \frac{e^{\alpha_N (1-\beta) \frac{N}{N-1} \varphi_N}}{|x|^{N\beta}} dx + o_\epsilon(1). \end{aligned}$$

Hence

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_{\epsilon, \gamma}|^N dx \geq \gamma \int_{B_R} \frac{e^{\alpha_N (1-\beta) \frac{N}{N-1} \varphi_N}}{|x|^{N\beta}} dx.$$

In view of (75), passing to the limit $R \rightarrow +\infty$, we obtain

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_{\epsilon, \gamma}|^N dx \geq \gamma. \quad (84)$$

Similarly we have

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} |\nabla (u_\epsilon - \gamma c_\epsilon)^+|^N dx \geq 1 - \gamma. \quad (85)$$

Noting that $\|u_\epsilon\|_{L^N(\mathbb{R}^N)} = o_\epsilon(1)$, we have

$$\int_{\mathbb{R}^N} |\nabla u_{\epsilon, \gamma}|^N dx + \int_{\mathbb{R}^N} |\nabla (u_\epsilon - \gamma c_\epsilon)^+|^N dx = \int_{\mathbb{R}^N} |\nabla u_\epsilon|^N dx = 1 + o_\epsilon(1). \quad (86)$$

Combining (84)-(86), we conclude the lemma. \square

Lemma 19. *We have*

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N, \epsilon} u_\epsilon^{N/(N-1)})}{|x|^{N\beta}} dx = \lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}}. \quad (87)$$

As a consequence, for any $\theta < N/(N-1)$, there holds $\lambda_\epsilon / c_\epsilon^\theta \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

Proof. Let $0 < \gamma < 1$ be fixed and $u_{\epsilon, \gamma}$ be defined as before. Applying the mean value theorem to the function $\zeta(N, t)$ and recalling (59), we have

$$\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)}) = \zeta(N-1, \xi_\epsilon) \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)} \leq \zeta(N-1, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)}) \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)},$$

where ξ_ϵ lies between $\beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)}$ and 0. Since $\zeta(N-1, t) = \zeta(N, t) + t^{N-2}/(N-2)!$ for all $t \geq 0$, it follows from the above inequality that

$$\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)}) \leq \zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)}) \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)} + \beta_{N, \epsilon}^{N-1} u_{\epsilon, \gamma}^N / (N-2)!. \quad (88)$$

It is easy to see that

$$\int_{\mathbb{R}^N} \frac{u_\epsilon^q}{|x|^{N\beta}} dx = o_\epsilon(1), \quad \forall q \geq N. \quad (89)$$

In view of Lemma 18, one can find some $p > 1$ such that

$$\limsup_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, p \beta_{N, \epsilon} u_{\epsilon, \gamma}^{N/(N-1)})}{|x|^{N\beta}} dx < \infty. \quad (90)$$

By the Hölder inequality and (60), one has

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}}) \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}}}{|x|^{N\beta}} dx \leq \left(\int_{\mathbb{R}^N} \frac{\zeta(N, p \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \right)^{1/p} \left(\int_{\mathbb{R}^N} \frac{(\beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}})^{p'}}{|x|^{N\beta}} dx \right)^{1/p'}, \quad (91)$$

where $1/p + 1/p' = 1$. Combining (88)-(91), one concludes

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = 0. \quad (92)$$

Since $u_\epsilon \rightarrow 0$ in $L_{\text{loc}}^q(\mathbb{R}^N)$ for any $q > 0$, we obtain

$$\begin{aligned} \int_{u_\epsilon > \gamma c_\epsilon} \frac{\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}})}{|x|^{N\beta}} dx &= \int_{u_\epsilon > \gamma c_\epsilon} \frac{e^{\beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx + o_\epsilon(1) \\ &\leq \frac{1}{\gamma^{\frac{N}{N-1}}} \int_{u_\epsilon > \gamma c_\epsilon} \frac{u_{\epsilon, \gamma}^{\frac{N}{N-1}}}{c_\epsilon^{\frac{N}{N-1}}} \frac{e^{\beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx + o_\epsilon(1) \\ &\leq \frac{1}{\gamma^{\frac{N}{N-1}}} \frac{\lambda_\epsilon}{c_\epsilon^{\frac{N}{N-1}}} + o_\epsilon(1). \end{aligned} \quad (93)$$

Combining (92) and (93), we have

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \leq \frac{1}{\gamma^{N/(N-1)}} \liminf_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}}.$$

Letting $\gamma \rightarrow 1$, we conclude

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N, \epsilon} u_{\epsilon, \gamma}^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \leq \liminf_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}}. \quad (94)$$

An obvious analog of (35) is

$$\limsup_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}} \leq \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx. \quad (95)$$

Combining (94) and (95), we obtain (87), which together with (68) implies that $\lambda_\epsilon/c_\epsilon^{N/(N-1)}$ has a positive lower bound. Then for any $\theta < N/(N-1)$, there holds

$$\lambda_\epsilon/c_\epsilon^\theta = c_\epsilon^{N/(N-1)-\theta} \lambda_\epsilon/c_\epsilon^{N/(N-1)} \rightarrow \infty.$$

This proves the second assertion of the lemma. \square

Lemma 20. $c_\epsilon^{\frac{1}{N-1}} u_\epsilon \rightarrow G$ in $C_{\text{loc}}^1(\mathbb{R}^N \setminus \{0\})$ and weakly in $W^{1,q}(\mathbb{R}^N)$ for any $1 < q < N$, where G is a distributional solution to

$$-\Delta_N G + \tau G^{N-1} = \delta_0 \quad \text{in } \mathbb{R}^N.$$

Moreover, $G \in W^{1,N}(\mathbb{R}^N \setminus B_r)$ for any $r > 0$ and G takes the form

$$G(x) = -\frac{N}{\alpha_N} \log |x| + A_0 + w(x),$$

where A_0 is a constant, and $w \in C^0(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ satisfies $w(x) = O(|x|^N \log^{N-1} |x|)$ as $|x| \rightarrow 0$.

Proof. Multiplying both sides of (67) by c_ϵ , we have

$$-\Delta_N(c_\epsilon^{\frac{1}{N-1}} u_\epsilon) + \tau c_\epsilon u_\epsilon^{N-1} = \frac{c_\epsilon u_\epsilon^{\frac{1}{N-1}}}{\lambda_\epsilon} \frac{\zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} \quad \text{in } \mathbb{R}^N.$$

Replacing Lemma 7 and Corollary 10 with Lemma 17 and Lemma 19 respectively in the proof of Lemma 11, we obtain for any $\phi \in C_0^1(\mathbb{R}^N)$,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{c_\epsilon u_\epsilon^{\frac{1}{N-1}}}{\lambda_\epsilon} \frac{\zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} \phi dx = \phi(0).$$

Since the remaining part of the proof is completely analogous to that of ([15], Proposition 3.7 and Lemma 3.8), we omit the details but refer the reader to [15]. \square

To estimate the supremum $\Lambda_{N,\beta,\tau}$, we need the following:

Lemma 21. Let $w_\epsilon \in W_0^{1,N}(B_r)$ satisfy $\int_{B_r} |\nabla w_\epsilon|^N dx \leq 1$, $w_\epsilon \rightharpoonup 0$ weakly in $W_0^{1,N}(B_r)$, and w_ϵ is nonnegative and radially symmetric. Then

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r} \frac{e^{\alpha_N(1-\beta)w_\epsilon^{N/(N-1)}} - 1}{|x|^{N\beta}} dx \leq \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} r^{N(1-\beta)} e^{\sum_{k=1}^{N-1} \frac{1}{k}}. \quad (96)$$

Proof. We first prove (96) for $r = 1$.

Denote $w_\epsilon(|x|) = w_\epsilon(x)$. Let $v_\epsilon(x) = (1 - \beta)^{(N-1)/N} w_\epsilon(|x|^{1/(1-\beta)})$. Then

$$\int_{B_1} |\nabla v_\epsilon|^N dx = \int_{B_1} |\nabla w_\epsilon|^N dx.$$

Clearly we can assume up to a subsequence, $v_\epsilon \rightharpoonup v_0$ weakly in $W_0^{1,N}(B_1)$, $v_\epsilon \rightarrow v_0$ strongly in $L^N(B_1)$, and $v_\epsilon \rightarrow v_0$ a.e. in B_1 . Also, we can assume $w_\epsilon \rightarrow 0$ a.e. in B_1 . Hence we conclude $v_0 = 0$ a.e. in B_1 . By a change of variable $t = s^{1/(1-\beta)}$, there holds

$$\begin{aligned} \int_{B_1} \frac{e^{\alpha_N(1-\beta)w_\epsilon^{\frac{N}{N-1}}} - 1}{|x|^{N\beta}} dx &= \int_0^1 \frac{e^{\alpha_N(1-\beta)w_\epsilon^{\frac{N}{N-1}}(t)} - 1}{t^{N\beta}} \omega_{N-1} t^{N-1} dt \\ &= \frac{1}{1-\beta} \int_0^1 (e^{\alpha_N(1-\beta)w_\epsilon^{\frac{N}{N-1}}(s^{1/(1-\beta)})} - 1) \omega_{N-1} s^{N-1} ds \\ &= \frac{1}{1-\beta} \int_0^1 (e^{\alpha_N v_\epsilon^{\frac{N}{N-1}}(s)} - 1) \omega_{N-1} s^{N-1} ds \\ &= \frac{1}{1-\beta} \int_{B_1} (e^{\alpha_N v_\epsilon^{\frac{N}{N-1}}} - 1) dx. \end{aligned}$$

This together with Lemma 3 implies that

$$\limsup_{\epsilon \rightarrow 0} \int_{B_1} \frac{e^{\alpha_N(1-\beta)w_\epsilon^{\frac{N}{N-1}}} - 1}{|x|^{N\beta}} dx \leq \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k}}. \quad (97)$$

We next prove (96) for the case of general r . Set $\tilde{w}_\epsilon(x) = w_\epsilon(rx)$ for $x \in B_1$. One can check that

$$\int_{B_1} |\nabla \tilde{w}_\epsilon|^N dx = \int_{B_r} |\nabla w_\epsilon|^N dx$$

and that

$$\int_{B_r} \frac{e^{\alpha_N(1-\beta)w_\epsilon^{\frac{N}{N-1}}} - 1}{|x|^{N\beta}} dx = r^{N(1-\beta)} \int_{B_1} \frac{e^{\alpha_N(1-\beta)\tilde{w}_\epsilon^{\frac{N}{N-1}}} - 1}{|x|^{N\beta}} dx.$$

This together with (97) gives the desired result. \square

By the equation (67) and $\|u_\epsilon\|_{1,\tau} = 1$, we have

$$\begin{aligned} \int_{B_r} |\nabla u_\epsilon|^N dx &= 1 - \int_{\mathbb{R}^N \setminus B_r} (|\nabla u_\epsilon|^N + \tau u_\epsilon^N) dx - \tau \int_{B_r} u_\epsilon^N dx \\ &= 1 - \int_{\mathbb{R}^N \setminus B_r} \frac{u_\epsilon^{\frac{N}{N-1}}}{\lambda_\epsilon} \frac{\zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \\ &\quad + \int_{\partial B_r} u_\epsilon |\nabla u_\epsilon|^{N-2} \frac{\partial u_\epsilon}{\partial r} d\sigma - \tau \int_{B_r} u_\epsilon^N dx. \end{aligned} \quad (98)$$

We estimate the right three terms on the above equation respectively. The first term can be calculated by

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_r} \frac{u_\epsilon^{\frac{N}{N-1}}}{\lambda_\epsilon} \frac{\zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx &= \frac{1}{c_\epsilon^{N/(N-1)}} \frac{c_\epsilon^{N/(N-1)}}{\lambda_\epsilon} \int_{\mathbb{R}^N \setminus B_r} u_\epsilon^{\frac{N}{N-1}} \frac{\zeta(N-1, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \\ &= \frac{o_\epsilon(1)}{c_\epsilon^{N/(N-1)}}. \end{aligned} \quad (99)$$

A straightforward calculation on the second term reads

$$\begin{aligned} \int_{\partial B_r} u_\epsilon |\nabla u_\epsilon|^{N-2} \frac{\partial u_\epsilon}{\partial r} d\sigma &= \frac{1}{c_\epsilon^{N/(N-1)}} \left(\int_{\partial B_r} G |\nabla G|^{N-2} \frac{\partial G}{\partial r} d\sigma + o_\epsilon(1) \right) \\ &= \frac{1}{c_\epsilon^{N/(N-1)}} \left(G(r) \int_{B_r} \Delta_N G dx + o_\epsilon(1) \right) \\ &= \frac{1}{c_\epsilon^{N/(N-1)}} \left(-G(r) + \tau G(r) \int_{B_r} G^{N-1} dx + o_\epsilon(1) \right), \end{aligned} \quad (100)$$

since G is a distributional solution of $-\Delta_N G + \tau G^{N-1} = \delta_0$. Concerning the third term, one has

$$\int_{B_r} u_\epsilon^N dx = \frac{1}{c_\epsilon^{N/(N-1)}} \left(\int_{B_r} G^N dx + o_\epsilon(1) \right). \quad (101)$$

Inserting (99)-(101) into (98) and noting that $G(x) = -\frac{N}{\alpha_N} \log |x| + A_0 + w(x)$, we conclude

$$\int_{B_r} |\nabla u_\epsilon|^N dx = 1 - \frac{1}{c_\epsilon^{N/(N-1)}} \left(\frac{N}{\alpha_N} \log \frac{1}{r} + A_0 + o_\epsilon(1) + o_r(1) \right). \quad (102)$$

Define $u_{\epsilon,r} = (u_\epsilon - u_\epsilon(r))^+$, the positive part of $u_\epsilon - u_\epsilon(r)$. Obviously $u_{\epsilon,r} \in W_0^{1,N}(B_r)$. It follows from Lemma 21 that

$$\limsup_{\epsilon \rightarrow 0} \int_{B_r} \frac{e^{\alpha_N(1-\beta)u_{\epsilon,r}^{N/(N-1)}/\tau_{\epsilon,r}} - 1}{|x|^{N\beta}} dx \leq \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} r^{N(1-\beta)} e^{\sum_{k=1}^{N-1} \frac{1}{k}}, \quad (103)$$

where $\tau_{\epsilon,r} = \|\nabla u_\epsilon\|_{L^N(B_r)}^{N/(N-1)}$. One can see from Lemma 17 that $u_\epsilon = c_\epsilon + o_\epsilon(1)$ on $B_{Rr_\epsilon^{1/(1-\beta)}}$. This together with Lemma 20 and (102) leads to that on $B_{Rr_\epsilon^{1/(1-\beta)}} \subset B_r$,

$$\begin{aligned} \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}} &\leq \alpha_N(1-\beta)(u_{\epsilon,r} + u_\epsilon(r))^{\frac{N}{N-1}} \\ &= \alpha_N(1-\beta)u_{\epsilon,r}^{\frac{N}{N-1}} + \frac{N}{N-1} \alpha_N(1-\beta)u_{\epsilon,r}^{\frac{1}{N-1}} u_\epsilon(r) + o_\epsilon(1) \\ &= \alpha_N(1-\beta)u_{\epsilon,r}^{\frac{N}{N-1}} + \frac{N}{N-1} \alpha_N(1-\beta)G(r) + o_\epsilon(1) \\ &= \alpha_N(1-\beta)u_{\epsilon,r}^{\frac{N}{N-1}} + \frac{N}{N-1} \alpha_N(1-\beta) \left(\frac{N}{\alpha_N} \log \frac{1}{r} + A_0 \right) + o_r(1) + o_\epsilon(1) \\ &= \alpha_N(1-\beta)u_{\epsilon,r}^{\frac{N}{N-1}} / \tau_{\epsilon,r} + N(1-\beta) \log \frac{1}{r} + \alpha_N(1-\beta)A_0 + o_r(1) + o_\epsilon(1). \end{aligned}$$

This together with (103) leads to

$$\begin{aligned}
\int_{B_{Rr_\epsilon}^{1/(1-\beta)}} \frac{e^{\beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}}} - 1}{|x|^{N\beta}} dx &\leq r^{-N(1-\beta)} e^{\alpha_N(1-\beta)A_0 + o(1)} \int_{B_{Rr_\epsilon}^{1/(1-\beta)}} \frac{e^{\alpha_N(1-\beta)u_{\epsilon,r}^{\frac{N}{N-1}}/\tau_{\epsilon,r}}}{|x|^{N\beta}} dx \\
&= r^{-N(1-\beta)} e^{\alpha_N(1-\beta)A_0 + o(1)} \int_{B_{Rr_\epsilon}^{1/(1-\beta)}} \frac{e^{\alpha_N(1-\beta)u_{\epsilon,r}^{\frac{N}{N-1}}/\tau_{\epsilon,r}} - 1}{|x|^{N\beta}} dx + o(1) \\
&\leq \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0} + o(1).
\end{aligned} \tag{104}$$

In view of (79), we obtain

$$\begin{aligned}
\int_{B_{Rr_\epsilon}^{1/(1-\beta)}} \frac{\zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx &= r_\epsilon^N \int_{B_R} \frac{e^{\beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}} (r_\epsilon^{\frac{1}{1-\beta}} y)}}{|y|^{N\beta}} dy + o_\epsilon(1) \\
&= \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}} \left(\int_{B_R} \frac{e^{\alpha_N(1-\beta) \frac{N}{N-1} \varphi_N(y)}}{|y|^{N\beta}} dy + o_\epsilon(1) \right) + o_\epsilon(1) \\
&= \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}} (1 + o_R(1) + o_\epsilon(1)) + o_\epsilon(1).
\end{aligned}$$

Therefore

$$\lim_{R \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \int_{B_{Rr_\epsilon}^{1/(1-\beta)}} \frac{\zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx = \lim_{\epsilon \rightarrow 0} \frac{\lambda_\epsilon}{c_\epsilon^{N/(N-1)}}. \tag{105}$$

Combining (104), (105) and (87), we conclude

$$\Lambda_{N,\beta,\tau} = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{\zeta(N, \beta_{N,\epsilon} u_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \leq \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0}. \tag{106}$$

3.5. Test function computation

We now construct test functions such that (106) does not hold. Precisely we construct a sequence of functions $\phi_\epsilon \in W^{1,N}(\mathbb{R}^N)$ satisfying $\|\phi_\epsilon\|_{1,\tau} = 1$ and

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)\phi_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx > \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0} \tag{107}$$

for sufficiently small $\epsilon > 0$. For this purpose we set

$$\phi_\epsilon(x) = \begin{cases} c + \frac{1}{c^{1/(N-1)}} \left(-\frac{N-1}{\alpha_N(1-\beta)} \log(1 + c_N(|x|/\epsilon)^{\frac{N}{N-1}(1-\beta)}) + b \right), & x \in \overline{B}_{R\epsilon} \\ \frac{G}{c^{1/(N-1)}}, & x \in \mathbb{R}^N \setminus B_{R\epsilon}, \end{cases}$$

where $c_N = \alpha_N/(N^{N/(N-1)}(1-\beta)^{1/(N-1)})$, G is given as in Lemma 20, $R = (-\log \epsilon)^{1/(1-\beta)}$, b and c are constants depending only on ϵ and β to be determined later. Note that $G \in W^{1,N}(\mathbb{R}^N \setminus B_r)$ for any $r > 0$. To ensure $\phi_\epsilon \in W^{1,N}(\mathbb{R}^N)$, we let

$$c + \frac{1}{c^{1/(N-1)}} \left(-\frac{N-1}{\alpha_N(1-\beta)} \log(1 + c_N R^{\frac{N}{N-1}(1-\beta)}) + b \right) = \frac{G(R\epsilon)}{c^{1/(N-1)}}.$$

By Lemma 20, we have $G(x) = -(N/\alpha_N) \log |x| + A_0 + w(x)$, where $w(x) = O(|x|^N \log^{N-1} |x|)$ as $|x| \rightarrow 0$. Then the above equality leads to

$$c^{\frac{N}{N-1}} = \frac{1}{\alpha_N(1-\beta)} \log \frac{\omega_{N-1}}{N(1-\beta)} + A_0 - b - \frac{N}{\alpha_N} \log \epsilon + O(R^{-\frac{N}{N-1}(1-\beta)}). \quad (108)$$

Now we calculate by the equation of G ,

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B_{R\epsilon}} (|\nabla \phi_\epsilon|^N + \tau \phi_\epsilon^N) dx &= \frac{1}{c^{\frac{N}{N-1}}} \int_{\mathbb{R}^N \setminus B_{R\epsilon}} (|\nabla G|^N + \tau G^N) dx \\ &= -\frac{1}{c^{\frac{N}{N-1}}} \int_{\partial B_{R\epsilon}} G |\nabla G|^{N-2} \frac{\partial G}{\partial \nu} d\sigma \\ &= \frac{1}{c^{\frac{N}{N-1}}} G(R\epsilon) \left(1 - \tau \int_{B_{R\epsilon}} G^{N-1} dx \right) \\ &= \frac{1}{c^{\frac{N}{N-1}}} \left(-\frac{N}{\alpha_N} \log(R\epsilon) + A_0 + O((R\epsilon)^N \log^N(R\epsilon)) \right). \end{aligned} \quad (109)$$

Note that for any $T > 0$, there holds

$$\begin{aligned} I_N(T) &\equiv \int_0^T \frac{t^{N-1}}{(1+t)^N} dt \\ &= \frac{1}{1-N} \int_0^T t^{N-1} d(1+t)^{1-N} \\ &= \frac{1}{1-N} \left(\frac{T}{1+T} \right)^{N-1} + \int_0^T \frac{t^{N-2}}{(1+t)^{N-1}} dt \\ &= \frac{1}{1-N} \left(\frac{T}{1+T} \right)^{N-1} + I_{N-1}(T). \end{aligned}$$

Since $I_1(T) = \log(1+T)$, we have by iteration

$$I_N(T) = \log(1+T) - \sum_{k=1}^{N-1} \frac{1}{k} \left(\frac{T}{1+T} \right)^k.$$

Hence, by a change of variables $t = c_N(r/\epsilon)^{N(1-\beta)/(N-1)}$, we obtain

$$\begin{aligned} \int_{B_{R\epsilon}} |\nabla \phi_\epsilon|^N dx &= \frac{1}{\omega_{N-1}^{\frac{1}{N-1}} c^{\frac{N}{N-1}}} \int_0^{R\epsilon} \frac{r^{\frac{N^2}{N-1}(1-\beta)-1}}{(r^{\frac{N}{N-1}(1-\beta)} + c_N^{-1} \epsilon^{\frac{N}{N-1}(1-\beta)})^N} dr \\ &= \frac{1}{\omega_{N-1}^{\frac{1}{N-1}} c^{\frac{N}{N-1}}} \frac{N-1}{N(1-\beta)} \int_0^{c_N R^{\frac{N}{N-1}(1-\beta)}} \frac{t^{N-1}}{(1+t)^N} dt \\ &= \frac{N-1}{\alpha_N(1-\beta) c^{\frac{N}{N-1}}} \left\{ \log \left(1 + c_N R^{\frac{N}{N-1}(1-\beta)} \right) - \sum_{k=1}^{N-1} \frac{1}{k} \left(\frac{c_N R^{\frac{N}{N-1}(1-\beta)}}{1 + c_N R^{\frac{N}{N-1}(1-\beta)}} \right)^k \right\} \\ &= \frac{1}{\alpha_N(1-\beta) c^{\frac{N}{N-1}}} \left\{ \log \frac{\omega_{N-1}}{N(1-\beta)} + N(1-\beta) \log R \right. \\ &\quad \left. - (N-1) \sum_{k=1}^{N-1} \frac{1}{k} + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}} \right) \right\}. \end{aligned} \quad (110)$$

Moreover, we require b to be bounded with respect to ϵ . It then follows from (108) that

$$\int_{B_{R\epsilon}} \phi_\epsilon^N dx = O((R\epsilon)^N (\log \epsilon)^{N-1}). \quad (111)$$

Combining (109)-(111), we obtain

$$\begin{aligned} \|\phi_\epsilon\|_{1,\tau}^N &= \frac{1}{c^{\frac{N}{N-1}}} \left(-\frac{N}{\alpha_N} \log \epsilon + A_0 - \frac{N-1}{\alpha_N(1-\beta)} \sum_{k=1}^{N-1} \frac{1}{k} + \frac{1}{\alpha_N(1-\beta)} \log \frac{\omega_{N-1}}{N(1-\beta)} \right. \\ &\quad \left. + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right) + O((R\epsilon)^N (\log \epsilon)^N) \right). \end{aligned}$$

Setting $\|\phi_\epsilon\|_{1,\tau} = 1$, we have

$$c^{\frac{N}{N-1}} = -\frac{N}{\alpha_N} \log \epsilon + A_0 - \frac{N-1}{\alpha_N(1-\beta)} \sum_{k=1}^{N-1} \frac{1}{k} + \frac{1}{\alpha_N(1-\beta)} \log \frac{\omega_{N-1}}{N(1-\beta)} + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right), \quad (112)$$

which together with (108) leads to

$$b = \frac{N-1}{\alpha_N(1-\beta)} \sum_{k=1}^{N-1} \frac{1}{k} + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right). \quad (113)$$

Denote

$$b_\epsilon(x) = -\frac{N-1}{\alpha_N(1-\beta)} \log(1 + c_N(|x|/\epsilon)^{\frac{N}{N-1}(1-\beta)}) + b. \quad (114)$$

Then $c^{-N/(N-1)} b_\epsilon(x) = O((\log \log \epsilon^{-1}) / \log \epsilon)$ uniformly in $x \in B_{R\epsilon}$, where $R = (\log \epsilon^{-1})^{1/(1-\beta)}$. We have by the Taylor formula of $(1+t)^{N/(N-1)}$ near $t=0$,

$$\begin{aligned} \phi_\epsilon^{\frac{N}{N-1}}(x) &= c^{\frac{N}{N-1}} \left(1 + c^{-\frac{N}{N-1}} b_\epsilon(x) \right)^{\frac{N}{N-1}} \\ &= c^{\frac{N}{N-1}} \left(1 + \frac{N}{N-1} c^{-\frac{N}{N-1}} b_\epsilon(x) + \frac{1}{2} \frac{N}{(N-1)^2} (1+\xi)^{\frac{2-N}{N-1}} (c^{-\frac{N}{N-1}} b_\epsilon(x))^2 \right) \\ &\geq c^{\frac{N}{N-1}} + \frac{N}{N-1} b_\epsilon(x), \end{aligned} \quad (115)$$

where ξ lies between $c^{-\frac{N}{N-1}} b_\epsilon(x)$ and 0. Inserting (112)-(114) into (115), we obtain for all $x \in B_{R\epsilon}$,

$$\begin{aligned} \alpha_N(1-\beta) \phi_\epsilon^{\frac{N}{N-1}}(x) &\geq -N(1-\beta) \log \epsilon + \alpha_N(1-\beta) A_0 + \sum_{k=1}^{N-1} \frac{1}{k} + \log \frac{\omega_{N-1}}{N(1-\beta)} \\ &\quad -N \log(1 + c_N(|x|/\epsilon)^{\frac{N}{N-1}(1-\beta)}) + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right). \end{aligned} \quad (116)$$

Also we have by a change of variables $t = c_N r^{\frac{N}{N-1}(1-\beta)}$ and integration by parts,

$$\begin{aligned}
\int_{B_R} \frac{1}{(1 + c_N |y|^{\frac{N}{N-1}(1-\beta)})^N |y|^{N\beta}} dy &= \int_0^R \frac{\omega_{N-1} r^{N-1-N\beta}}{(1 + c_N r^{\frac{N}{N-1}(1-\beta)})^N} dr \\
&= -\frac{t^{N-2}}{(1+t)^{N-1}} \Big|_0^{c_N R^{\frac{N}{N-1}(1-\beta)}} + \int_0^{c_N R^{\frac{N}{N-1}(1-\beta)}} \frac{(N-2)t^{N-3}}{(1+t)^{N-1}} dt \\
&= \int_0^{c_N R^{\frac{N}{N-1}(1-\beta)}} \frac{1}{(1+t)^2} dt + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right) \\
&= 1 + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right). \tag{117}
\end{aligned}$$

Combining (116) and (117), we obtain

$$\begin{aligned}
\int_{B_{R\epsilon}} \frac{\zeta(N, \alpha_N(1-\beta)\phi_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx &= \int_{B_{R\epsilon}} \frac{e^{\alpha_N(1-\beta)\phi_\epsilon^{\frac{N}{N-1}}}}{|x|^{N\beta}} dx + O(c^{\frac{N(N-2)}{N-1}}(R\epsilon)^{N(1-\beta)}) \\
&\geq \frac{\omega_{N-1}}{N(1-\beta)\epsilon^{N(1-\beta)}} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0 + O(R^{-\frac{N}{N-1}(1-\beta)})} \\
&\quad \times \int_{B_{R\epsilon}} \frac{1}{(1 + c_N(|x|/\epsilon)^{\frac{N}{N-1}(1-\beta)})^N |x|^{N\beta}} dx + O(c^{\frac{N(N-2)}{N-1}}(R\epsilon)^{N(1-\beta)}) \\
&= \frac{\omega_{N-1}}{N(1-\beta)} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0 + O(R^{-\frac{N}{N-1}(1-\beta)})} \\
&\quad \times \int_{B_R} \frac{1}{(1 + c_N |y|^{\frac{N}{N-1}(1-\beta)})^N |y|^{N\beta}} dy + O(c^{\frac{N(N-2)}{N-1}}(R\epsilon)^{N(1-\beta)}) \\
&= \frac{\omega_{N-1}}{N(1-\beta)} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0} + O\left(\frac{1}{R^{\frac{N}{N-1}(1-\beta)}}\right). \tag{118}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_{R\epsilon}} \frac{\zeta(N, \alpha_N(1-\beta)\phi_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx &\geq \frac{\alpha_N^{N-1}(1-\beta)^{N-1}}{(N-1)!c^{\frac{N}{N-1}}} \int_{\mathbb{R}^N \setminus B_{R\epsilon}} \frac{G^N}{|x|^{N\beta}} dx \\
&= \frac{\alpha_N^{N-1}(1-\beta)^{N-1}}{(N-1)!c^{\frac{N}{N-1}}} \left(\int_{\mathbb{R}^N} \frac{G^N}{|x|^{N\beta}} dx + o_\epsilon(1) \right). \tag{119}
\end{aligned}$$

Combining (118), (119) and noting that $R^{-\frac{N}{N-1}(1-\beta)}c^{\frac{N}{N-1}} = o_\epsilon(1)$, we have

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1-\beta)\phi_\epsilon^{\frac{N}{N-1}})}{|x|^{N\beta}} dx \geq \frac{\omega_{N-1} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0}}{N(1-\beta)} + \frac{(\alpha_N(1-\beta))^{N-1}}{(N-1)!c^{\frac{N}{N-1}}} \left(\int_{\mathbb{R}^N} \frac{G^N}{|x|^{N\beta}} dx + o_\epsilon(1) \right).$$

Therefore we conclude (107) for sufficiently small $\epsilon > 0$.

3.6. Completion of the proof of Theorem 2

Under the assumption that $c_\epsilon \rightarrow +\infty$, there holds (106). While it follows from (107) that

$$\Lambda_{N\beta, \tau} > \frac{1}{1-\beta} \frac{\omega_{N-1}}{N} e^{\sum_{k=1}^{N-1} \frac{1}{k} + \alpha_N(1-\beta)A_0}.$$

This contradicts (106) and implies that c_ϵ must be bounded. Then applying Theorem 14 to the equation (67), we get the desired extremal function. \square

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